Analysis Qualifying Review, Saturday September 8, 2012

Morning Session, 9:00–12:00 Noon

(1.) Let $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$, and let $f : \bar{\mathbb{H}} \to \mathbb{C}$ be a bounded continuous function , which is analytic in $\mathbb{H}$. Prove that for any $z = x + iy \in \mathbb{H}$

$$f(z) = \frac{x}{\pi} \int_{-\infty}^{\infty} \frac{f(it)\,dt}{x^2 + (t-y)^2}.$$

(2.) Let $f : \mathbb{D} \to \mathbb{D}, f(z) = \sum_{n=0}^{\infty} a_n z^n$, be a bounded analytic function

a.) Prove that for any $r < 1$

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2\,dt.$$

b.) Show that the series $\sum_{n=0}^{\infty} |a_n|^2$ converges.

(3.) Let f be a function which is analytic on the wedge

$$W = \{z \in \mathbb{C} \,|\, \mathrm{Re}(z) > 0, -\frac{\pi}{6} < \arg(z) < \frac{\pi}{6}\},$$

which is bounded on W, and verifies for all $r > 0$

$$\lim_{\theta \to \pm\frac{\pi}{6}} f(re^{i\theta}) := \varphi(r) \in \mathbb{R}.$$

Show that f must be real and constant.

Hint: Consider using Schwarz reflection.

(4.) Evaluate

$$\int_0^{\infty} \frac{\ln x\,dx}{(x-1)\sqrt{x}}.$$

(5.) Let $\Omega \subset \mathbb{C}$ be a bounded, simply connected domain in $\mathbb{C}$. Let z_0 and z_1 be two distinct points of Ω. If φ_1 and φ_2 are two one-to-one and onto analytic maps from Ω onto itself, and $\varphi_1(z_i) = \varphi_2(z_i), i = 1, 2$, show that $\varphi_1 \equiv \varphi_2$ on Ω.

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Afternoon Session, 2:00–5:00 PM

(1.) Let $f \in L_1((0,1))$, and define $g : (0,1) \to \mathbb{R}$ by

$$g(x) = \int_x^1 \frac{f(t)}{t}\, dt.$$

Prove that $g \in L_1((0,1))$.

(2.) Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let $F : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^2$ function with second derivative $F'' > 0$. Let $f \in L_1(\mu)$ be real-valued. Prove *Jensen's inequality*:

$$F(\frac{1}{\mu(X)} \int f\, d\mu) \leq \frac{1}{\mu(X)} \int F(f) d\mu.$$

(3.) Let $f, g_1, g_2 \ldots \in L_1(\mathbb{R})$ be non-negative functions. Assume that $g_n \to f$ a.e. and

$$\int_{\mathbb{R}} g_n\, dm = \int_{\mathbb{R}} f\, dm.$$

Prove that for any measurable set $A \subseteq \mathbb{R}$

$$\int_A g_n\, dm \to \int_A f\, dm.$$

(4.) Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Let $\{f_n\}_{n=1}^{\infty} \subset L_2(\mu)$ be a sequence of functions such that $\|f_n\|_2 \leq 1$.

a.) Prove that if $f_n \to 0$ in measure, then $f_n \to 0$ in $L_1(\mu)$.

b.) If $f_n \to 0$ in measure, does it necessarily follow that $f_n \to 0$ in $L_2(\mu)$?

(5.) Let $F \subset \mathbb{R}$ be a closed set, and define the distance from $x \in \mathbb{R}$ to F by

$$d(x, F) = \inf_{y \in F} |x - y|.$$

Prove that

$$\lim_{x \to y} \frac{d(x, F)}{|x - y|} = 0$$

for a.e. $y \in F$.

Hint: Consider Lebesgue points of F.