Analysis Qualifying Review. January 12, 2013

Morning Session, 9:00 am - 12:00 noon

1. Let f be a continuously differentiable function on $\mathbb{R}$. Assume there exist constants $a, b \in \mathbb{R}$ such that

$$f(x) \to a, \quad f'(x) \to b \quad \text{as } x \to \infty.$$

Prove or give a counterexample: b must be zero.

2. Using techniques of real analysis (as opposed to complex analysis) show that

$$\lim_{R\to\infty} \int_0^R \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

Hint: compute $\int_0^\infty e^{-xt}\, dt$ first.

3. Let f be a measurable function on a measure space (X, μ), where μ is a finite measure. Suppose there are constants $K > 0$ and $p > 1$ such that

$$\mu\big\{x \in X : |f(x)| > M\big\} < \frac{K}{M^p} \quad \text{for all } M > 0.$$

Prove that f is integrable.

4. Let E be a measurable subset of $[0, 1]$. Assume there is a constant $\alpha > 0$ such that

$$m(E \cap I) \geq \alpha\, m(I) \quad \text{for all intervals } I \subset [0, 1].$$

(Here $m(\cdot)$ denotes Lebesgue measure.) Prove that $m(E) = 1$.

5. Let $(f_n)_{n=1}^\infty$ be a sequence of non-negative measurable functions on a measure space (X, μ), where μ is a finite measure. Assume that f_n converges almost everywhere to an integrable function f.

(a) Show by example that in general $\lim_n \int f_n\, d\mu$ may be infinite.

(b) Suppose $\lim_n \int f_n\, d\mu = \int f\, d\mu$. Prove that $f_n \to f$ in L_1, that is

$$\lim_n \int |f_n - f|\, d\mu = 0.$$

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Afternoon Session, 2:00–5:00

(1) Let Δ denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$, let H denote the right half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and let f be an analytic function mapping H into Δ and satisfying $f(5) = 0$. Based on the given information, what are you able to say about $f'(5)$?

(2) Let $f = u + iv$ be an entire function with the property that $(u^2)_{xx} + (u^2)_{yy}$ vanishes identically along the real axis. (The subscripts denote partial derivatives.) What can you conclude about f?

(3) Find all solutions of $\cos z = 1 + 100z^2$ in the unit disk $|z| < 1$.

(4) Let S denote the strip $\{z = x + iy : 0 \leq y \leq 1\}$. Suppose that f is analytic in a neighborhood of S and satisfies $|f(x + iy)| \leq \frac{C}{1+x^2}$ on S for some constant C.

(a) Show that

$$I(t, y) \stackrel{\text{def}}{=} \int_{\mathbb{R}} f(x + iy)e^{-ixt}\, dx$$

is a well-defined bounded continuous function of $(t, y) \in \mathbb{R} \times [0, 1]$.

(b) Under what conditions will $|I(t, 1)|$ be bigger than $|I(t, 0)|$?

(5) Construct a function $f(z)$ analytic for $0 < |z| < 1$ so that $e^{f(z)}$ has a pole at $z = 0$, or else explain why no such function exists.