Analysis Qualifying Review, January 11, 2014

Morning Session, 9:00 am–noon

Notation: m denotes Lebesgue measure

(1) Prove or disprove: If E is an open subset of $\mathbb{R}$ with $m(E) = 1$ then there is a finite union of intervals F containing E with $m(F) < 1.1$.

(2) Let $f \in L_1 \cap L_4$ (on some measure space). Prove that the function

$$\begin{aligned} [1,4] &\to \mathbb{R} \\ p &\mapsto \|f\|_p \end{aligned}$$

is continuous.

(3) Find all $q \geq 1$, such that $f(x^2) \in L_q((0,1), m)$ for any $f(x) \in L_4((0,1), m)$.

(4) Let

$$\begin{aligned} E &\subset \{(x,y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq x\} \\ E_x &= \{y \mid (x,y) \in E\} \\ E^y &= \{x \mid (x,y) \in E\} \end{aligned}$$

and assume that $m(E_x) \geq x^3$ for any $x \in [0,1]$.

(a) Prove that there exists $y \in [0,1]$ such that $m(E^y) \geq \dfrac{1}{4}$.

(b) Prove that there exists $y \in [0,1]$ such that $m(E_y) \geq c$, where $c > 1/4$ is a constant independent of E. Give an explicit value of c.

(5) Let $E \subset [0,1]$ be a measurable set, $m(E) \geq \dfrac{99}{100}$. Prove that there exists $x \in [0,1]$ such that for any $r \in (0,1)$,

$$m\big(E \cap (x-r, x+r)\big) \geq \frac{r}{4}.$$

Remark: One approach to this problem involves the Hardy-Littlewood maximal inequality.

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Afternoon Session, 2:00–5:00 pm

Notation: $\mathbb{D} = \{z : \ |z| < 1\}$

(1) Find all entire functions $f(z)$ with the property that $g(z) \stackrel{\text{def}}{=} f(2z + \bar{z})$ is also entire.

(2) How many zeros does the polynomial

$$p(z) = z^8 + 10z^3 - 50z + 1$$

have in the right half-plane?

(3) Does there exists an analytic function f with an essential singularity at 0 such that $f(z) + 2f(z^2)$ has a removable singularity?

(4) Let $\{f_n : \mathbb{D} \to \mathbb{C}\}_{n=1}^{\infty}$ be a sequence of analytic functions such that $f_n(0) = 0$ for all $n \in \mathbb{N}$, and $\mathrm{Re} f_n(z) \to 0$ uniformly on compact sets. Prove that $\mathrm{Im} f_n(z) \to 0$ uniformly on compact sets.

(5) Use complex integration methods to compute $\displaystyle\int_0^\infty \frac{x^t}{(x+1)(x+2)}\, dx$, where $t \in (0, 1)$.