Analysis Qualifying Review, May 8, 2014

Morning Session, 9:00 am–noon

Notation: $\mathbb{D} = \{z : \ |z| < 1\}$

(1) Prove that there exists an analytic function $f : \mathbb{D} \to \mathbb{D}$ such that $f(1/2) = f(-1/2)$ and $f'(z) \neq 0$ for all $z \in \mathbb{D}$.

(2) Let f be a polynomial such that
$$|f(z)| \leq 1 - |z|^2 + |z|^{1000}$$
for all $z \in \mathbb{C}$. Prove that $|f(0)| \leq 0.2$.

(3) Let $f_k : \mathbb{D} \to \mathbb{C}$ be a normal family of analytic functions and let $h_k : \mathbb{D} \to \mathbb{D}$ be analytic functions satisfying $h_k(0) = 0$. Prove that the functions
$$g_k(z) = f_k\left(h_k(z)\right)$$
form a normal family.

(4) Let
$$\Omega_1 = \mathbb{C} \setminus \left(\{0\} \cup \left\{ \frac{1}{n} \ : \ n \in \mathbb{N} \right\} \right)$$
and
$$\Omega_2 = \mathbb{C} \setminus \{z \ : \ \operatorname{Im} z = 0, |\operatorname{Re} z| \geq 1\}.$$
Construct a non-constant analytic function $f : \Omega_1 \to \Omega_2$ or show that this is impossible.

(5) Let $f(z)$ be the branch of $\sqrt{z^2 - 1}$ on $|z| > 1$ satisfying $\lim_{z\to\infty} \frac{f(z)}{z} = 1$.

(a) Determine the coefficients $\alpha, \beta\gamma, \delta, \varepsilon$ in the Laurent expansion
$$f(z) = \alpha z + \beta + \gamma z^{-1} + \delta z^{-2} + \varepsilon z^{-3} + \dots.$$

(b) Compute $\displaystyle\int_{|z|=2} (5 + 6z + 7z^2) f(z)\, dz.$

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Afternoon Session, 2:00–5:00 pm

Notation: m denotes Lebesgue measure

(1) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Prove that the set of points where f is differentiable is measurable.

(2) Let $f_1, f_2, \ldots, f, g$ be measurable functions on a measure space $(X, \mathcal{A}, \mu)$. Assume that $f_n \to f$ in measure and $f_n \le g$ a.e. Prove that $f \le g$ a.e.

(3) Let $q_1, q_2, q_2, \ldots$ be an enumeration of $\mathbb{Q} \cap [0, 1]$ and let $r, t \in (0, 1)$. Consider the set

$$E \stackrel{\text{def}}{=} \left\{ x \in [0,1] : \sum_{j=1}^{\infty} t^j |x - q_j|^{-r} < \infty \right\}.$$

(a) Show that $E \neq [0, 1] \setminus \mathbb{Q}$.

(b) Show that $m([0, 1] \setminus E) = 0$.

(4) For $f \in L^1(\mathbb{R}, m)$ let $Tf(x) = \int\limits_{x-1}^{x+1} f\, dm.$

(a) Show that if $Tf = 4f$ a.e. then $f = 0$ a.e.

(b) Does the conclusion of (a) still hold if we only assume that f is integrable on each bounded interval in $\mathbb{R}$?

(5) Prove that the sequence

$$f_n(x) = n^{1/2} \exp\left(-\frac{n^2 x^2}{x + 1} \right)$$

converges in $L_p([0, +\infty), m)$ for $1 \le p < 2$ and diverges for $p \ge 2$.