Analysis Qualifying Review, September 6, 2014

Morning Session, 9:00 am–noon

Notation: m denotes Lebesgue measure

(1) Construct a measurable subset A of $(0,1)$ such that $m(A) < 1$ and $m\big(A \cap (a,b)\big) > 0$ for any $(a,b) \subset (0,1)$.

(2) Let $\{f_k(x)\}$ be a sequence of nonnegative measurable functions on E and $m(E) < \infty$. Show that $\{f_k(x)\}$ converges in measure to 0 if and only if

$$\lim_{k\to\infty} \int_E \frac{f_k(x)}{1 + f_k(x)}\, dx = 0$$

(3) Let $1 \le p < \infty$, $f \in L^p(\mathbb{R}^n)$. Let

$$f_*(\lambda) = m(\{x : |f(x)| > \lambda\}), \quad \lambda > 0$$

Show that

(a) $p \int_0^\infty \lambda^{p-1} f_*(\lambda)\, d\lambda = \int |f(x)|^p\, dx$

(b) $\lim_{\lambda\to\infty} \lambda^p f_*(\lambda) = 0, \qquad \lim_{\lambda\to 0} \lambda^p f_*(\lambda) = 0$

(4) Let $K = \{f : (0, +\infty) \to \mathbb{R} \mid \int_0^\infty f^4(x)\, dx \le 1\}$. Evaluate

$$\sup_{f\in K} \int_0^\infty f^3(x) e^{-x}\, dx.$$

(5) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $\int_{\mathbb{R}} |f(x)|\, dx < \infty$. Show that the sequence

$$h_n(x) = \frac{1}{n} \sum_{k=1}^{n} f\left(x + \frac{k}{n}\right)$$

converges in $L_1(\mathbb{R})$.

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Afternoon Session, 2 pm – 5 pm

(1) Assume that 0 is an isolated singularity of an analytic function $f \neq 0$. Determine the type of the singularity if

$$\sum_{n=1}^{\infty} |f(1/n)|^{1/n} < +\infty.$$

(2) Show that for $x, y \in \mathbb{R}$,

$$|y| \leq |\sin(x+iy)| \leq e^{|y|}$$

(3) Let $a \in (0,1)$. Find

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\, dx$$

(4) Find a conformal mapping of domain $\bar{\mathbb{C}} \setminus \{x + xi, 1 \leq x \leq 2\}$ to the upper half plane. Here $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.
It is enough to represent the mapping as a composition of several conformal mappings.

(5) Denote by $\mathbb{D}$ the unit disc: $\mathbb{D} = \{z : \ |z| < 1\}$. Let $\{f_k : \mathbb{D} \to \mathbb{C}\}_{k \in \mathbb{N}}$ be a normal family. Prove that the functions

$$g_k(z) = f_k\left(e^{ik} z\right),\ k \in \mathbb{N}$$

form a normal family.