Analysis Qualifying Review, January 10, 2015

Morning Session, 9:00 am–noon

Notation: m denotes Lebesgue measure, m^* is the Lebesgue outer measure

(1) Let f be a nonnegative measurable function on $(0,1)$. Assume that there is a constant c, such that

$$\int_0^1 (f(x))^n \, dx = c, \qquad n = 1, 2, \ldots$$

Show that there is a measurable set $E \subset (0,1)$, such that

$$f(x) = \chi_E(x), \qquad \text{for a.e. } x \in (0,1).$$

(2) Let f be locally integrable on $\mathbb{R}^n$, $1 < p < \infty$. Show that the following are equivalent:
(1) $f \in L^p(\mathbb{R}^n)$;
(2) there exist $M > 0$, such that for any finite collection of mutually disjoint measurable sets $E_1, E_2, \ldots, E_k$, with $0 < m(E_i) < \infty$ for $1 \le i \le k$,

$$\sum_{i=1}^{k} \left(\frac{1}{m(E_i)}\right)^{p-1} \left|\int_{E_i} f(x)\, dx\right|^p \le M$$

(3) Let $f : \mathbb{R} \times [0,1] \to \mathbb{R}$ be a measurable function such that for any $y \in [0,1]$,

$$\int_{\mathbb{R}} f^2(x,y)\, dx \le 1.$$

Prove that there exists a sequence $x_n \to +\infty$ such that

$$\int_0^1 f(x_n, y)\, dy \to 0.$$

(4) Let $E_k \subset [a,b]$, $k \in \mathbb{N}$ be measurable sets, and there exists $\delta > 0$ such that $m(E_k) \ge \delta$ for all k. Assume that $a_k \in \mathbb{R}$ satisfies

$$\sum_{k=1}^{\infty} |a_k| \chi_{E_k}(x) < \infty \qquad \text{for a.e. } x \in [a,b].$$

Show that

$$\sum_{k=1}^{\infty} |a_k| < \infty.$$

(Hint: one of the possible approaches uses Egorov's theorem.)

(5) Let $A, B \subset \mathbb{R}^d$. Assume $A \cup B$ is measurable, and $m(A \cup B) < \infty$. If

$$m(A \cup B) = m^*(A) + m^*(B)$$

Show that A and B are measurable.

(Hint: prove first that for any set A, there a measurable set U, with $A \subset U$, such that $m^*(A) = m(U)$.)

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Afternoon Session, 2 pm – 5 pm

(1) Is there a function f, analytic at the origin, taking values $f(\frac{1}{2n}) = f(\frac{1}{2n-1}) = \frac{1}{2n}$, for $n \in \mathbb{N}$?

(2) Find and classify the singularities of the function $f(z) = \sin\left(\frac{1}{\sin z}\right)$.

(3) Let f be an entire function such that $f(z) \in \mathbb{R}$, whenever $z \in \mathbb{R}$. Assume that $|f(z)| \leq 1$ on the boundary of the rectangle with the vertices $-1, 1, 1+i, -1+i$. Prove that $|f'(\frac{i}{10})| \leq \frac{10}{9}$.

(4) Suppose Ω is a bounded region, f is holomorphic on Ω and

$$\limsup_{n\to\infty} |f(z_n)| \leq M$$

for every sequence z_n in Ω which converges to a boundary point of Ω. Prove that $|f(z)| \leq M$ for all $z \in \Omega$.

(5) Let $f \in C^1(\mathbb{R})$ be a function with compact support.

(a) Show that for any $x \in \mathbb{R}$,

$$\text{p.v.}\int_{-\infty}^{\infty} \frac{f(\zeta)}{\zeta - x}\, d\zeta := \lim_{\epsilon\to 0} \int_{|\zeta - x| > \epsilon} \frac{f(\zeta)}{\zeta - x}\, d\zeta \quad \text{exists;}$$

(b) Prove that

$$\lim_{y\to 0+} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(\zeta)}{\zeta - (x+iy)}\, d\zeta = \frac{1}{2} f(x) + \frac{1}{2\pi i} \cdot \text{p.v.}\int_{-\infty}^{\infty} \frac{f(\zeta)}{\zeta - x}\, d\zeta.$$