Analysis Qualifying Review, May 7, 2015

Morning Session, 9:00 am–noon

(1) Let $E \subset \mathbb{R}^1$. Show that the characteristic function $\chi_E(x)$ is the limit of a sequence of continuous functions if and only if E is both F_σ and G_δ.

(2) Let $\{g_n\}$ be a sequence of measurable functions on $[a, b]$, satisfying
(a) $|g_n(x)| \leq M$, $a.e. x \in [a, b]$;
(b) For every $c \in [a, b]$, $\lim_{n\to\infty} \int_a^c g_n(x)\, dx = 0$.
Show that for any $f \in L^1[a, b]$,

$$\lim_{n\to\infty} \int_a^b f(x) g_n(x)\, dx = 0.$$

(3) Let $f_k(x)$, $k = 1, 2 \ldots$ be increasing functions on $[a, b]$. Assume

$$\sum_{k=1}^{\infty} f_k(x)$$

is convergent on $[a, b]$. Show that

$$\Big(\sum_{k=1}^{\infty} f_k(x)\Big)' = \sum_{k=1}^{\infty} f_k'(x), \qquad a.e. x \in [a, b]$$

(4) (a) Assume that $f \in L^\infty(\mathbb{R})$, and f is continuous at 0. Show that

$$\lim_{n\to\infty} \int \frac{n}{\pi(1 + (nx)^2)} f(x)\, dx = f(0).$$

(b) Assume that $f \in L^\infty(\mathbb{R})$. Show that

$$\lim_{n\to\infty} \int \frac{n}{\pi(1 + n^2(x - y)^2)} f(y)\, dy = f(x) \qquad a.e. x \in \mathbb{R}$$

(Hint: $\int \frac{1}{\pi(1+y^2)}\, dy = 1$.)

(5) Let $\{f_n\}$ be a sequence of functions in $L^p(\mathbb{R}^n)$, $1 < p < \infty$, which converge almost everywhere to a function $f \in L^p(\mathbb{R}^n)$, and suppose that there is a constant M such that $\|f_n\|_p \leq M$ for all n. Show that for every $g \in L^q(\mathbb{R}^n)$, q the conjugate of p,

$$\int fg = \lim_{n\to\infty} \int f_n g$$

Is the statement true for $p = 1$?
(Hint: you may want to use Egorov's Theorem.)

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Afternoon Session, 2 pm – 5 pm

Notation:

- $\mathbb{D} = \{z : |z| < 1\}$
- $\#S =$ cardinality of S

(1) Construct an explicit analytic bijection from

$$\{z \in \mathbb{C} : |z| > 1, z \text{ not real and positive}\}$$

to

$$\{z \in \mathbb{C} : \operatorname{Re} z > 0\}.$$

(You may write your mapping as a composition of simpler explicit mappings.)

(2) Let $A = \{z \in \mathbb{C} : 5 \le |z| \le 10\}$.

(a) Prove or disprove: there is a function f analytic on a neighborhood of A and satisfying $|f(z)| < 1$ for $|z| = 10$, $|f(z)| > 1000$ for $|z| = 5$.

(b) Prove or disprove: there is a function f analytic on a neighborhood of A and satisfying $\operatorname{Re} f(z) < 1$ for $|z| = 10$, $\operatorname{Re} f(z) > 1000$ for $|z| = 5$.

(3) For f analytic on $\mathbb{D}$ let

$$\sigma(f) = \sup\left\{\#f^{-1}(w) : w \in \mathbb{C}\right\}.$$

(a) Prove or disprove: There exists a sequence f_n of functions analytic on $\mathbb{D}$ converging uniformly on compact sets of $\mathbb{D}$ to a limit function f with all $\sigma(f_n) = 3$ but $\sigma(f) = 4$.

(b) Prove or disprove: There exists a sequence f_n of functions analytic on $\mathbb{D}$ converging uniformly on compact sets of $\mathbb{D}$ to a limit function f with all $\sigma(f_n) = 4$ but $\sigma(f) = 3$.

(4) Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function defined on a neighborhood of $\overline{\mathbb{D}}$ and satisfying

- $f(\overline{\mathbb{D}}) \subset \mathbb{D}$;
- $f(0) = 0$.

Let $f^{\circ n} = \overbrace{f \circ \cdots \circ f}^{n \text{ times}}$.

Show that $f^{\circ n}(z) \to 0$ for $z \in \mathbb{D}$.

(5) Suppose $\{f_n\}$ is a uniformly bounded sequence of analytic functions on a domain Ω such that $\{f_n(z)\}$ converges for every $z \in \Omega$.

(a) Show that the convergence is uniform on every compact subset of Ω.

(b) Must $\{f_n'\}$ converge uniformly on every compact subset of Ω? Prove or disprove.