Chapter 6
SPHERICAL GEOMETRY

So far we have studied finite and discrete geometries, i.e., geometries in which the main transformation group is either finite or discrete. In this chapter, we begin our study of infinite continuous geometries with spherical geometry, the geometry ($\mathbb{S}^2$:O(3)) of the isometry group of the two-dimensional sphere, which is in fact the subgroup of all isometries of $\mathbb{R}^3$ that map the origin to itself; O(3) is called the *orthogonal group* in linear algebra courses.

But first we list the classical continuous geometries that will be studied in this course. Some of them may be familiar to the reader, others will be new.

6.1. A list of classical continuous geometries

Here we merely list, for future reference, several very classical geometries whose transformation groups are "continuous" rather than finite or discrete. We will not make the intuitively clear notion of continuous transformation group precise (this would involve defining the so-called *topological groups* or even *Lie groups*), because we will not study this notion in the general case: it is not needed in this introductory course. The material of this section is not used in the rest of the present chapter, so the reader who wants to learn about spherical geometry without delay can immediately go on to Sect. 6.3.

6.1.1. *Finite-dimensional vector spaces* over the field of real numbers are actually geometries in the sense of Klein (the main definition of Chapter 1). From that point of view, they can be written as

$$\boxed{(\mathbb{V}^n : \mathrm{GL}(n))},$$

where $\mathbb{V}^n$ denotes the n-dimensional vector space over $\mathbb{R}$ and GL(n) is the *general linear group*, i.e., the group of all nondegenerate linear transformations of $\mathbb{V}^n$ to itself.

The subgeometries of $(\mathbb{V}^n : \mathrm{GL}(n))$ obtained by replacing the group GL(n) by its subgroup O(n) (consisting of orthogonal transformations) is called the *n-dimensional orthonormed vector space* and denoted

$$\boxed{(\mathbb{V}^n : \mathrm{O}(n))}.$$

These "geometries" are rather algebraic and are usually studied in linear algebra courses. We assume that the reader has some background in linear algebra and remembers the first basic definitions and facts of the theory.

6.1.2. *Affine spaces* are, informally speaking, finite-dimensional vector spaces "without a fixed origin". This means that their transformation groups $\mathrm{Aff}(n)$ contain, besides $\mathrm{GL}(n)$, all parallel translations of the space (i.e., transformations of the space obtained by adding a fixed vector to all its elements). We denote the corresponding geometry by

$$\boxed{(\mathbb{V}^n : \mathrm{Aff}(n))} \quad \text{or} \quad \boxed{(\mathbb{R}^n : \mathrm{Aff}(n))},$$

the later notation indicating that the elements of the space are now regarded as *points*, i.e., the endpoints of the vectors (issuing from the origin) rather than the vectors themselves. This is a more geometric notion than that of vector space, but is also usually studied in linear algebra courses.

6.1.3. *Euclidean spaces* are geometries that we denote

$$\boxed{(\mathbb{R}^n : \mathrm{Sym}(\mathbb{R}^n))};$$

here $\mathrm{Sym}(\mathbb{R}^n)$ is the isometry group of Euclidean space $\mathbb{R}^n$, i.e., the group of distance-preserving transformations of $\mathbb{R}^n$. This group has, as a subgroup, the *orthogonal group* $\mathrm{O}(n)$ that consists of isometries leaving the origin fixed (the group $\mathrm{O}(n)$ should be familiar from the linear algebra course), but also contains the subgroup of parallel translations.

We assume that, for $n = 2, 3$, the reader knows Euclidean geometry from school (of course it was introduced differently, usually via some modification of Euclid's axioms) and is familiar with the structure of the isometry groups of Euclidean space for $n = 2, 3$.

The reader who feels uncomfortable with elementary Euclidean plane and space geometry can consult Appendix I. A rigorous axiomatic approach to Euclidean geometry in dimensions $d = 2, 3$ (based on Hilbert's axioms) appears in Appendix III.

Note that the transformation groups of these three geometries (vector spaces, affine and Euclidean spaces) act on the same space ($\mathbb{R}^n$ and $\mathbb{V}^n$ can be naturally identified), but the geometries that they determine are different, because the four groups $\mathrm{GL}(n)$, $\mathrm{O}(n)$, $\mathrm{Aff}(n)$, $\mathrm{Sym}(\mathbb{R}^3)$ are different. The corresponding geometries will not be studied in this course: traditionally, this is done in linear algebra courses, and we have listed them here only to draw a complete picture of classical geometries.

Our list continues with three more classical geometries that we will study, at least in small dimensions (mostly in dimension 2).

6.1.4. *Hyperbolic spaces* $\mathbb{H}^n$ (called *Lobachevsky spaces* in Russia) are "spaces of constant negative curvature" (you will learn what this means much later, in differential geometry courses) with transformation group the isometry group of the hyperbolic space (i.e., the group of transformations preserving the "hyperbolic distance"). We will only study the hyperbolic space of dimension $n = 2$, i.e., the hyperbolic plane. Three models of $\mathbb{H}^2$ will be studied, in particular, the *Poincaré disk model*,

$$\boxed{(\mathbb{H}^2 : \mathcal{M})};$$

here $\mathbb{H}^2 := \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$ is the open unit disk and $\mathcal{M}$ is the group of *Möbius transformations* (the definition appears in Chapter 7) that take the disk to itself.

We will also study two other models of hyperbolic plane geometry (the *half-plane model*, also due to Poincaré, and the *Cayley–Klein model*). A special chapter describes how attempts to prove Euclid's Fifth Postulate led to the appearance of hyperbolic plane geometry and the dramatic history of its creation by Gauss, Lobachevsky and Bolyai.

6.1.5. *Elliptic spaces* $\mathbb{E}\mathrm{ll}^n$ are "spaces of constant positive curvature" (what this means is explained in differential geometry courses). We will only study the two-dimensional case, i.e., the elliptic plane, in the present chapter after we are done with *spherical geometry*, which is the main topic of this chapter, but can also be regarded as the principal building block of elliptic plane geometry.

6.1.6. *Projective spaces* $\mathbb{R}P^n$ are obtained from affine spaces by "adding points at infinity" in a certain way, and taking, for the transformation group, a group of linear transformations on the so-called "homogeneous coordinates" of points $(x_1 : \cdots : x_n : x_{n+1}) \in \mathbb{R}P^n$. We can write this geometry as

$$\boxed{(\mathbb{R}P^n : \mathrm{Proj}(n))}.$$

For arbitrary n, projective geometry is usually studied in linear algebra courses. We will study the *projective plane* $\mathbb{R}P^2$ in this course, and only have a quick glance at projective space $\mathbb{R}P^3$ (see Chapter 12).

6.2. Some basic facts from Euclidean plane geometry

Here we list several fundamental facts of Euclidean plane geometry (including modern formulations of some of Euclid's postulates) in order to compare and contrast them with the corresponding facts of spherical, elliptic, and hyperbolic geometry.

I. *There exist a unique (straight) line passing through two given distinct points.*

II. *There exists a unique perpendicular to a given line passing through a given point.* (A *perpendicular* to a given line is a line forming four equal angles, called *right angles*, with the given one.)

III. *There exists a unique circle of given center and given radius.*

IV. *Given a point on a line and any positive number, there exist exactly two points on the line whose distance from the given point is equal to the given number.*

V. *There exists a unique parallel to a given line passing through a given point not on the given line.* (A *parallel* to a given line is a line without common points with the given one.) This is the modern version of Euclid's fifth postulate, sometimes described as the single most important and controversial scientific statement of all time.

VI. *The parameters of a triangle ABC, namely the angles α, β, γ at the vertices A, B, C and the sides a, b, c opposite to these vertices, satisfy the following formulas.*

(i) *Angle sum formula:* $\alpha + \beta + \gamma = \pi$.

(ii) *Sine formula:*

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

(ii) *Cosine formula:* $c^2 = a^2 + b^2 - 2ab\cos\gamma$.

6.3. Lines, distances, angles, polars, and perpendiculars on $\mathbb{S}^2$

Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$:

$$\mathbb{S}^2 := \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\};$$

our present aim is to study the geometry $(\mathbb{S}^2 : \mathrm{O}(3))$, where $\mathrm{O}(3)$ is the orthogonal group (i.e., the group of isometries of $\mathbb{R}^3$ leaving the origin in place).

6.3.1. *Basic definitions.* By a *line* on the sphere we mean a great circle, i.e., the intersection of $\mathbb{S}^2$ with a plane passing through the sphere's center. For example, the equator of the sphere, as well as any meridian, is a line. The *angle* between two lines is defined as the dihedral angle (measured in radians) between the two planes containing the lines. For example, the angle between the equator and any meridian is $\pi/2$. The *distance* between two points A and B is defined as the measure (in radians) of the angle AOB. Thus the distance between the North and South Poles is π, the distance between the South Pole and any point on the equator is $\pi/2$.

Obviously, *the transformation group $O(3)$ preserves distances between points.* It can also be shown (we omit the proof) that, conversely, O(3) can be characterized as the group of distance-preserving transformations of the sphere (distance being understood in the spherical sense, i.e., as explained above).

6.3.2. *Poles, polars, perpendiculars, circles.* Let us look at the analogs in spherical geometry of the Euclidean postulates.

I$_S$**.** *There exist a unique line passing through two given distinct points, except when the two points are antipodal, in which case there are infinitely many.* All the meridians joining the two poles give an example of the exceptional situation.

II$_S$**.** *There exists a unique perpendicular to a given line passing through a given point, except when the point lies at the intersection of the perpendicular constructed from the center O of the sphere to the plane in which the line lies, in which case there are infinitely many such perpendiculars.* The exceptional

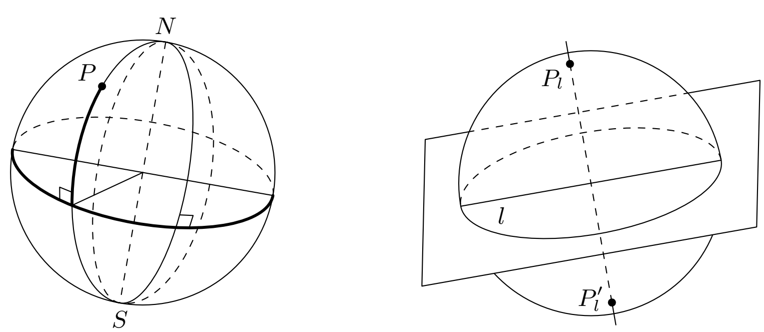

Figure 6.1. Perpendiculars, poles, and polars

situation is exemplified by the equator and, say, the North Pole: all the meridians (which all pass through the pole) are perpendicular to the equator.

More generally, the *polar* of a point P is the (spherical) line obtained by cutting the sphere by the plane passing through O and perpendicular to the (Euclidean!) straight line PO. Conversely, given a (spherical) line l, the *poles* of that line are the two antipodal points P_l and P_l' for which the (Euclidean) line P_lP_l' is perpendicular to the plane determined by l. The assertion II_S may now be restated as follows: *there exists a unique perpendicular to a given line passing through a given point, except when the point is a pole of that line, in which case all the lines passing through the pole are perpendicular to the given line.*

$\mathbf{III}_S$. *There exists a unique circle of given center C and given radius ρ, provided* $0 < \rho < \pi$. It is defined as the set of points whose (spherical) distance from C is equal to ρ. It is easy to see that any (spherical) circle is actually a Euclidean circle, namely the one obtained as the intersection of the sphere with the plane perpendicular to the Euclidean line OC and passing through the point I on that line such that $OI = \cos\rho$. Note that the radius of the Euclidean circle will be less than ρ.

Given a spherical circle of center C and radius ρ, note that it can be regarded as the circle of radius $\pi - \rho$ and center C', where C' is the antipode of C. Further, note that the longest circle centered at C is the polar of the point C; its radius is $\pi/2$.

$\mathbf{IV}_S$. Given a point on a line and any positive number, *there exist exactly two points on the line whose distance from the given point is equal to the given number, provided the number is less than* π.

$\mathbf{V}_S$. Any two lines intersect in two *antipodal* points, i.e., in two points symmetric with respect to the center of the sphere $\mathbb{S}^2$. Therefore *there are no parallel lines in spherical geometry*. If two points A, B are not antipodal, then there is only one line joining them and one shortest line segment with endpoints at A and B. For opposite points, there is an infinity of lines joining them (for the North and South poles, these lines are the meridians).

6.3.3. *Lines as shortest paths.* It is proved in differential geometry courses that *spherical lines are geodesics*, i.e., they are the shortest paths between two points. To do this, one defines the length of a curve as a curvilinear integral and uses the calculus of variations to show that the curve (on the sphere) of minimal length joining two given points is indeed the arc of the great circle containing these points.

6.4. Biangles and triangles in $\mathbb{S}^2$

6.4.1. *Biangles.* Two lines l and m on the sphere intersect in two (antipodal) points P and P' and divide the sphere into four domains; each of them is called a *biangle*, it is bounded by two halves of the lines l and m, called its *sides*, and has two *vertices* (the points P and P'). The four domains form two congruent pairs; two biangles from a congruent pair touch each other at the common vertices P and P', and have the same angle at P and P'. The main parameter of a biangle is the measure α of the angle between the lines that determine it; if $\alpha \neq \pi/2$, the two biangles not congruent to the biangle of measure α are called *complementary*, their angle is $\pi - \alpha$. Note that the angle measure α determines the corresponding biangle up to an isometry of the sphere.

6.4.2. *Areas of figures in the sphere.* In order to correctly measure areas of figures on the plane, on the sphere, or on other surfaces, one must define what an area is, specify what figures are measurable (i.e., possess an area), and devise methods for computing areas. For the Euclidean plane, there are several approaches to area: many readers have probably heard of the theory of *Jordan measure*; more advanced readers may have studied *Lebesgue measure*; readers who have taken multivariable calculus courses know that areas may be computed by means of *double integrals*.

In this book, we will not develop a rigorous measure theory for the geometries that we study. In this subsection, we merely sketch an axiomatic approach for determining areas of spherical figures; this approach is similar to Jordan measure theory in the Euclidean plane. The theory says that there is a family of sets in $\mathbb{S}^2$, called *measurable*, satisfying the following axioms.

(i) *Invariance.* Two congruent measurable figures have the same area.

(ii) *Normalization.* The whole sphere is measurable and its area is 4π.

(iii) *Countable additivity.* If a measurable figure F is the union of a countable family of measurable figures $\{F_i\}$ without common interior points, then its area is equal to the sum of areas of the figures F_i.

An obvious consequence of these axioms is that the area of the North hemisphere is 2π, while each of the triangles obtained by dividing the hemisphere into four equal parts is of area $\pi/2$.

6.4.3. *Area of the biangle.* From the axioms formulated in the previous subsection, it is easy to deduce that the area $S_{\pi/2}$ of a biangle with angle measure $\pi/2$ is π. Indeed, the sphere is covered by four such non-overlapping biangles, which are congruent to each other; they have the same area by (i), the sum of their areas is that of the sphere by (iii), and the latter is 4π by (ii), whence $S_{\pi/2} = (4\pi)/4 = \pi$.

For the case in which the angle measure α of a biangle is a rational multiple of π, a similar argument shows that

$$\boxed{S_\alpha = 2\alpha}. \tag{6.1}$$

This formula is actually true for any α, but for the case in which π/α is irrational, its proof requires a passage to the limit based on an additional "continuity axiom" that we have not explicitly stated. We therefore omit the proof, but will use the above formula for all values of α in what follows.

6.4.4. *Area of the triangle.* Let A, B, C be three distinct points of $\mathbb{S}^2$, no two of which are opposite. The union of the shortest line segments joining the points A and B, B and C, C and A is called the *triangle* ABC. For a triangle ABC, we always denote by α, β, γ the measure of the angles at A, B, C respectively and by a, b, c the lengths of the sides opposite to A, B, C (recall that the length a of BC is equal to the measure of the angle BOC in $\mathbb{R}^3$).

6.4.5. Theorem. *The area S_{ABC} of a spherical triangle with angles α, β, γ is equal to*

$$\boxed{S_{ABC} = \alpha + \beta + \gamma - \pi}.$$

Proof. There are 12 spherical biangles formed by pairs of lines AB, BC, CA. Choose six of them, namely those that contain triangle ABC or the antipodal triangle $A_1B_1C_1$ formed by the three points antipodal to A, B, C. Denote their areas by

$$S_I, S_I', S_{II}, S_{II}', S_{III}, S_{III}'.$$

Each point of the triangles ABC and $A_1B_1C_1$ is covered by exactly three of the chosen six biangles, while the other points of the sphere are covered by exactly one such biangle (we ignore the points on the lines). Therefore, using relation (6.1), we can write

$$\begin{aligned} 4\pi &= S_I + S_I' + S_{II} + S_{II}' + S_{III} + S_{III}' - 2S_{ABC} - 2S_{A_1B_1C_1} \\ &= 2\alpha + 2\beta + 2\gamma + 2\alpha + 2\beta + 2\gamma - 2S_{ABC} - 2S_{A_1B_1C_1} \\ &= 4(\alpha + \beta + \gamma) - 4S_{ABC}, \end{aligned}$$

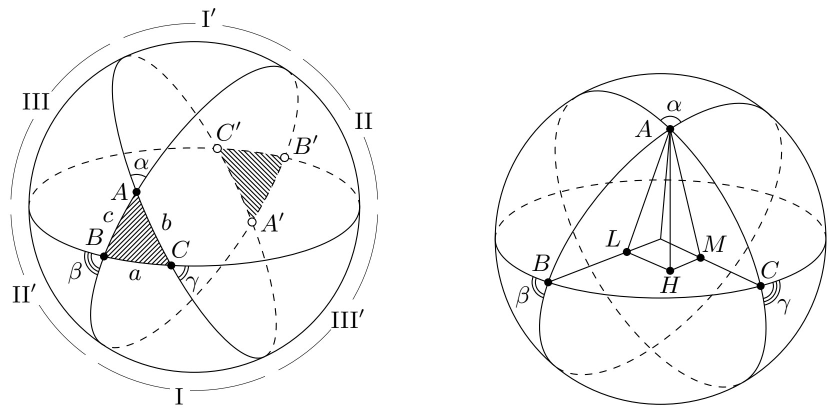

Figure 6.2. Area and sine theorem for the triangle

because the two triangles ABC $A_1B_1C_1$ have the same area (since they are congruent). Clearly, the previous formula implies the required equality. □

This theorem has the following fundamental consequence.

6.4.6. Corollary. *The sum of angles of any triangle is more than π.*

The analog of the sine formula for the Euclidean triangle is the following statement about spherical triangles.

6.4.7. Theorem. (The spherical sine theorem.)

$$\boxed{\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}.}$$

In order to establish this formula, we will use the following statement, sometimes called the "theorem on the three perpendiculars".

6.4.8. Lemma. *Let $A \in \mathbb{R}^3$ be a point outside a plane $\mathcal{P}$, let K be its perpendicular projection on $\mathcal{P}$ and let L be its perpendicular projection on a line l contained in $\mathcal{P}$. Then KL is perpendicular to l.*

Proof of the lemma. The line l is perpendicular to the plane AKL because it is perpendicular to two nonparallel lines of AKL, namely to AL and AK (to the latter since AK is orthogonal to any line in $\mathcal{P}$). Therefore l is perpendicular to any line of the plane AKL, and in particular to LK. □

Proof of the theorem. Let H be the projection of A on the plane ABC, let L and M be the projections of A on the lines OB and OC. Then by the lemma, L and M coincide with the projections of H on OB and OC. Therefore,

$$AH = LA \sin\beta = \sin c \sin\beta, \quad AH = MA \sin\gamma = \sin b \sin\gamma.$$

Thus, $\sin b : \sin\beta = \sin c : \sin\gamma$. Similarily, by projecting C on the plane AOB and arguing as above, we obtain $\sin b : \sin\beta = \sin a : \sin\alpha$. This immediately implies the required equality. □

6.5. Other theorems about triangles.

In this section, we state a few more theorems about spherical triangles. Their proofs are relegated to the exercises appearing at the end of this chapter.

6.5.1. Theorem. (The first cosine theorem.)

$$\boxed{\cos a = \cos b \cos c + \sin b \sin c \cos\alpha}$$

6.5.2. Theorem. (The second cosine theorem.)

$$\boxed{\cos\alpha + \cos\beta\cos\gamma = \sin\beta\sin\gamma\cos c}$$

6.5.3. Corollary. (Analog of the Pythagoras theorem.) *If triangle ABC has a right angle at C, then*

$$\boxed{\cos c = \cos a \cos b}.$$

6.5.4. Theorem. *The medians of any triangle intersect at a single point.*

6.5.5. Theorem. *The altitudes of any triangle intersect at a single point.*

6.6. Coxeter triangles on the sphere $\mathbb{S}^2$

We will not develop the theory of tilings on the sphere $\mathbb{S}^2$ and Coxeter geometry on the sphere in full generality, but only consider *Coxeter triangles*, i.e., spherical triangles all of whose angles are of the form π/m, $m = 2, 3, \ldots$. It follows from Theorem 6.4.5 that any spherical Coxeter triangle $(\pi/p, \pi/q, \pi/r)$, N copies of which cover the sphere, must satisfy the Diophantine equation

$$N/p + N/q + N/r = N + 4.$$

The transformation group of the corresponding Coxeter geometry is finite, and so Theorem 3.2.6 tells us what group it has to be: it must be either one of the dihedral groups, or the tetrahedral, hexahedral, or dodecahedral group. The dihedral groups yield an obvious infinite series of tilings, one of which is shown in Figure 6.3.

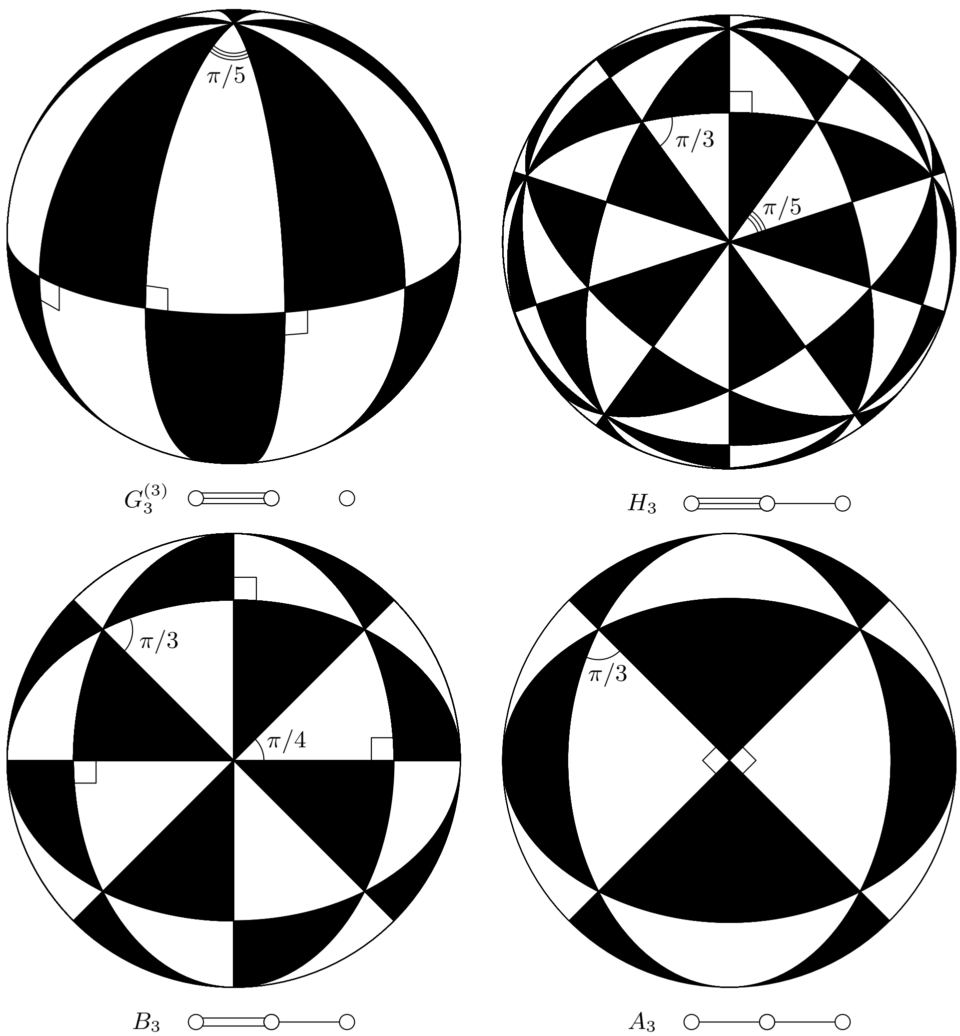

Figure 6.3. Four Coxeter tilings of the sphere

The three other groups yield three possibilities for N: $N = 24$, 48, 120, and we easily find the corresponding values of (p, q, r) in each of the three cases. Finally, the solutions of our Diophantine equation are:

$$(2,3,3),(2,3,4),(2,3,5),(2,2,n) \quad \text{for} \quad n=2,3,\ldots.$$

The corresponding tilings of the sphere (and their Coxeter schemes) are shown in Figure 6.3.

6.7. Two-dimensional elliptic geometry

6.7.1. Spherical geometry is closely related to the *elliptic geometry* invented by Riemann. Elliptic geometry is obtained from spherical geometry by "identifying opposite points of $\mathbb{S}^2$". The precise definition can be stated as follows. Consider the set $\mathbb{E}\mathrm{ll}^2$ whose elements are pairs of antipodal points $(x,-x)$ on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. The group $O(3)$ acts on this set (because isometries of $\mathbb{S}^2$ take antipodal pairs of points to antipodal pairs), thus defining a geometry in the sense of Klein $(\mathbb{E}\mathrm{ll}^2 : \mathrm{O}(3))$, which we call *two-dimensional elliptic geometry.*

Lines in elliptic geometry are defined as great circles of the sphere $\mathbb{S}^2$, angles and distances are defined as in spherical geometry, and the trigonometry of triangles in elliptic geometry is the same as in spherical geometry. More generally, one can say that elliptic geometry is locally the same as spherical, but these geometries are drastically different globally. In particular, in elliptic geometry

- *one and only line passes through any two distinct points*;
- *for a given line and any given point* (*except one, called the pole of that line*) *there exists a unique perpendicular to that line passing through the point.*

The relationship between the two geometries is best expressed by the following statement, which yields simple proofs of the statements about elliptic geometry made above,

6.7.2. Theorem. *There exists a surjective morphism*

$$D : (\mathbb{S}^2 : \mathrm{O}(3)) \to (\mathbb{E}\mathrm{ll}^2 : \mathrm{O}(3)),$$

of spherical geometry onto elliptic geometry which is a local isomorphism (*in the sense that any domain contained in a half-sphere is mapped bijectively and isometrically onto its image*).

Proof. The map D is the obvious one: $D : x \mapsto (x,-x)$, while the homomorphism of the transformation groups is the identity isomorphism. All the assertions of the theorem are immediate. $\square$

As we noted above, globally the two geometries are very different. Being metric spaces, they are topological spaces (in the metric topology) which are not even homeomorphic: one is a two-sided surface ($\mathbb{S}^2$), the other ($\mathbb{R}P^2$) is one-sided (it contains a Möbius strip).

6.8. Problems

In all the problems below a, b, c are the sides and α, β, γ are the opposite angles of a spherical triangle. The radius of the sphere is $R = 1$.

6.1. Prove the first cosine theorem on the sphere $\mathbb{S}^2$:

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha.$$

6.2. Prove the second cosine theorem on the sphere $\mathbb{S}^2$:

$$\cos \alpha + \cos \beta \cos \gamma = \sin \beta \sin \gamma \cos a.$$

6.3. Prove that $a + b + c < 2\pi$.

6.4. Does the Pythagorean theorem hold in spherical geometry? Prove the analogs of that theorem stated in Corollary 6.5.3.

6.5. Does the Moscow–New York flight fly over Spain? Over Greenland? Check your answer by stretching a thin string between Moscow and NY on a globe.

6.6. Find the infimum and the supremum of the sum of the angles of an equilateral triangle on the sphere.

6.7. The city A is located at the distance 1000km from the cities B and C, the trajectories of the flights from A to B and from A to C are perpendicular to each other. Estimate the distance between B and C. (You can take the radius of the Earth equal to 6400km)

6.8*. Find the area of the spherical disk of radius r (i.e., the domain bounded by a spherical circle of radius r).

6.9. Find fundamental domains for the actions of the isometry groups of the tetrahedron, the cube, the dodecahedron, and the icosahedron on the 2-sphere and indicate the number of their images under the corresponding group action.

6.10. Prove that any spherical triangle has a circumscribed and an inscribed circle.

6.11. Prove that the medians of a spherical triangle intersect at one point.

6.12. Prove that the altitudes of a spherical triangle always intersect at one point.

6.13. Suppose that the medians and the altitudes of a spherical triangle interest at the points M and A respectively. Can it happen that $M = A$?