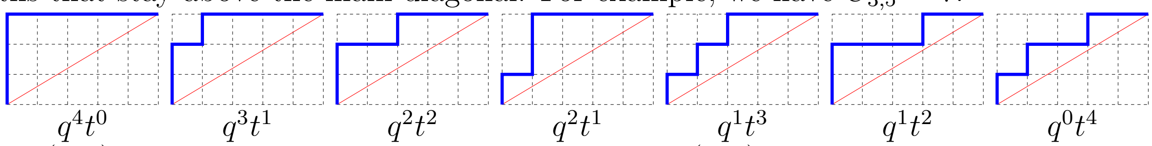
Michigan Research Experience for Graduates
Proposed dates: June 28 – July 2.

1. Rational Catalan numbers

Let $a, b \geq 1$ be positive integers satisfying $\gcd(a, b) = 1$. The *rational Catalan number* $C(a, b)$ is given by

$$C_{a,b} := \frac{1}{a+b}\binom{a+b}{a}.$$

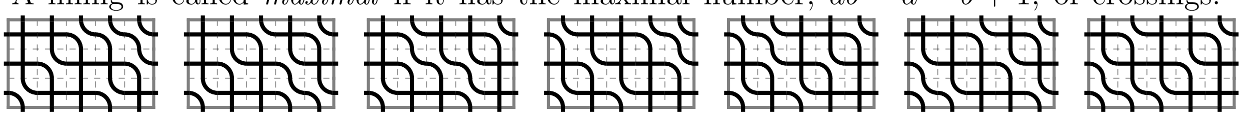
The Catalan numbers are given by $C_n := C_{n,n+1}$. Like the more famous Catalan numbers, rational Catalan numbers have many combinatorial interpretations. Here are some:
(a) $C_{a,b}$ is equal to the number of *rational Dyck paths* in a $a \times b$ rectangle: those lattice paths that stay above the main diagonal. For example, we have $C_{3,5} = 7$.



(b) $C(a, b)$ is equal to the number of simultaneous (a, b)-cores: partitions λ such that neither an a-ribbon nor a b-ribbon can be removed. [And] [AHJ]
(c) rational dissections and rational noncrossing partitions [ARW].

One aim of this project is to gain some familiarity with rational Catalan combinatorics.

In [GL], a new class of objects called *maximal* (a, b)*-Deograms* were shown to be counted by $C(a, b)$. These are wiring diagrams D in a $a \times b$ rectangle, where every box is filled with either a crossing or an elbow. The wires go from the NW to the SE, and the order in which the wires enter is the same as the order in which the wires exit. The wires are labeled $1, 2, \ldots, a + b$ in order SW to NE. They must satisfy the *distinguished* condition: for any elbow in D, the label of its SW wire is less than the label of its NE wire. In other words, once two wires have crossed an odd number of times, they cannot form an elbow. A filling is called *maximal* if it has the maximal number, $ab - a - b + 1$, of crossings.



Problem 1. *Find a bijection between maximal* (a, b)*-Deograms and* (a, b)*-rational Dyck paths.*

2. q, t-rational Catalan numbers

There is a q, t-analogue of $C_{a,b}$, the q, t-rational Catalan number [Hag]. It is defined as

$$C_{a,b}(q, t) := \sum_P q^{\text{area}(P)} t^{\text{dinv}(P)}, \tag{1}$$

where the sum is over (a, b)-rational Dyck paths P, and area(P) is the number of unit squares fully contained between P and the diagonal, and dinv(P) is the number of pairs (h, v) satisfying the following conditions: h is a horizontal step of P, v is a vertical step of P that appears to the right of h, and there exists a line of slope a/b (parallel to the diagonal) intersecting both h and v. For example,

$$C_{3,5}(q,t) = q^4 + q^3t + q^2t^2 + q^2t + qt^3 + qt^2 + t^4. \quad (2)$$

Problem 2. *Find statistics u, v so that $C_{a,b}(q,t) = \sum_D q^{u(D)}t^{v(D)}$, summed over maximal (a,b)-Deograms.*

3. Other interpretations

Rational Catalan numbers and their q,t-analogues have numerous interpretations in algebra and geometry, appearing in Macdonald theory, diagonal harmonics, torus knot homology, compactified Jacobians, positroid varieties, Hilbert schemes of points in the plane, Cherednik algebras, ...

References

[And] Jaclyn Anderson. Partitions which are simultaneously t_1- and t_2-core. Discrete Math., 248 (2002), 237– 243.

[AHJ] Drew Armstrong, Christopher R. H. Hanusa, Brant C. Jones. Results and conjectures on simultaneous core partitions. arXiv:1308.0572.

[ARW] Drew Armstrong, Brendon Rhoades, Nathan Williams, Rational associahedra and noncrossing partitions. arXiv:1305.7286.

[GL] Pavel Galashin and Thomas Lam. Positroids, knots, and q,t-Catalan numbers.

[Hag] J. Haglund. The q,t-Catalan Numbers and the Space of Diagonal Harmonics. https://www2.math.upenn.edu/ jhaglund/books/qtcat.pdf

[LW] Nicholas A. Loehr and Gregory S. Warrington. A continuous family of partition statistics equidistributed with length. J. Combin. Theory Ser. A, 116 (2009), 379–403.