Derangements and the p-adic incomplete gamma function

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Motivation

- Combinatorics: counts of permutations and permutation-related objects, e.g.

$$\#(\text{permutations on } [n]) = n!$$

- Number theory: special functions, modular forms, e.g. $\Gamma(s) = \int_0^\infty t^s e^{-t} \frac{dt}{t}$

COMBINATORICS $\leftrightarrow$ NUMBER THEORY

- Upshot: Gamma function identity $n! = \Gamma(n+1)$ extends factorial $n! : \mathbb{N} \to \mathbb{N}$ to complex-valued function $x! : \mathbb{C} \to \mathbb{C}$

p-adic numbers

A p-adic integer is a number of the form $a_0 + a_1 p + a_2 p^2 + a_3 p^3 + \cdots, \; a_i \in \mathbb{Z}$.

The topology on p-adic numbers is induced by the metric

$$|p^n a| \leq \frac{1}{p^n} \quad \text{if } a \in \mathbb{Z}.$$

$\mathbb{Z}_p$ denotes the set of p-adic integers, and $\mathbb{Q}_p$ the set of p-adic rationals.

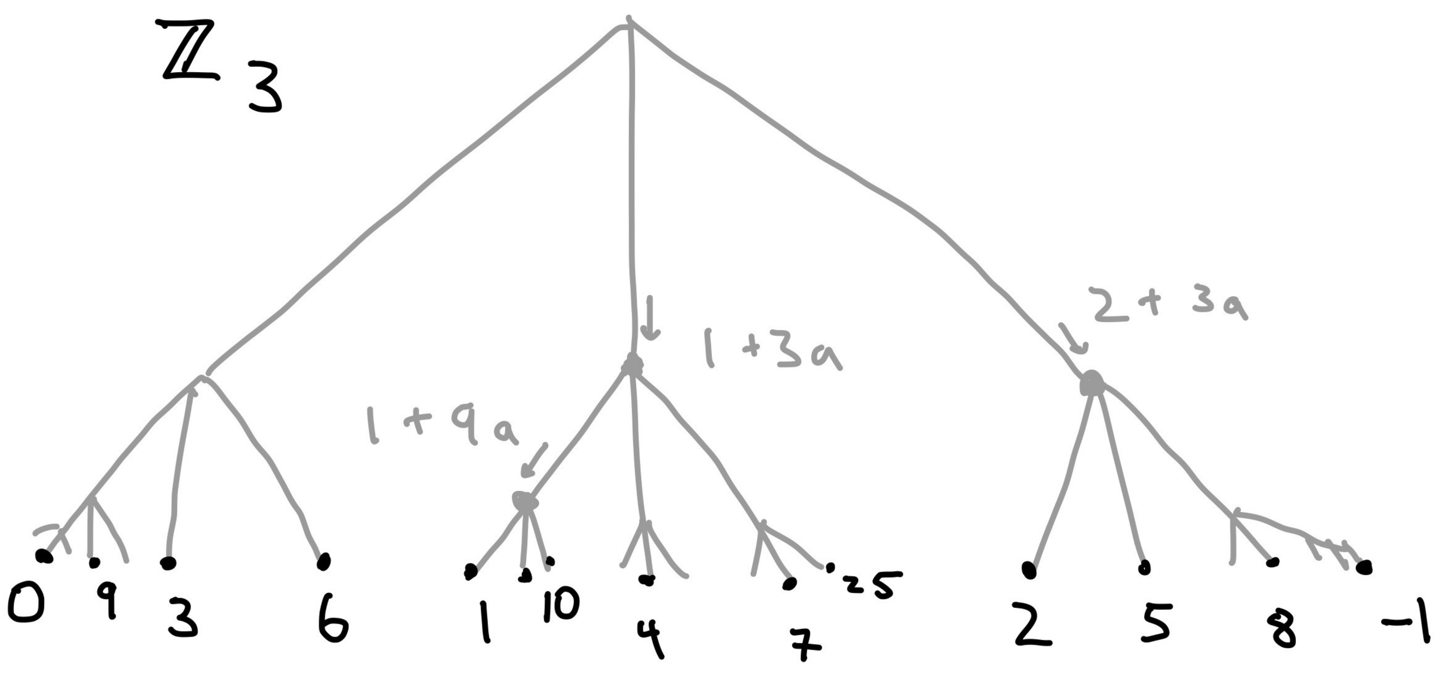

Figure 1. 3-adic integers with their topology.

Derangements

A derangement is a permutation on $[n]$ with no fixed points. By inclusion-exclusion, the number of derangements on n elements is $d(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!$.

n	0	1	2	3	4	5	6	7	8	9	10
$d(n)$	1	0	1	2	9	44	265	1854	14833	133496	14684570

Table 1. Number of derangements on n elements.

- Problem: What is value of $d(\infty)$? Or $d(-1)$? What "patterns" appear in $d(n)$?

A function f is pseudo-polynomial if $a \equiv b \mod n$ implies $f(a) \equiv f(b) \mod n$.

Theorem (Hall, 1971)

If $a \equiv b \mod n$ then $(-1)^a d(a) \equiv (-1)^b d(b) \mod n$.

Note that $(-1)^n d(n) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (n-k)! = \sum_{k=0}^{\infty} (-1)^k \binom{n}{k} k!$.

p-adic continuity

Problem: How to decide if function $f : \mathbb{N} \to \mathbb{Q}$ is p-adic continuous?

- Ex: $3n^2 + 5n + 1$ is p-adic continuous for every p.
- Ex: $(-1)^n$ is 2-adic continuous, but not 3-adic continuous.

The Mahler coefficients for a function $f : \mathbb{N} \to \mathbb{Q}$ are the constants c_k such that

$$f(x) = c_0 + c_1 \binom{x}{1} + c_2 \binom{x}{2} + c_3 \binom{x}{3} + \cdots.$$

- Ex: $3n^2 + 5n + 1 = 1 + 8\binom{n}{1} + 6\binom{n}{2}$.
- Ex: $(-1)^n = 1 - 2\binom{n}{1} + 4\binom{n}{2} - 8\binom{n}{3} + \cdots$, $c_k = (-1)^k 2^k$.

Example: Is $f(n) = 10^n$ p-adic continuous for any p?

Theorem (Mahler, 1958)

The function $f : \mathbb{N} \to \mathbb{Q}_p$ is p-adic continuous if and only if $|c_k|_p \to 0$ as $k \to \infty$, where c_k are Mahler coefficients of f.

Mahler coefficients can be found using finite differences

$$c_0 = f(0), \quad c_1 = \Delta f(0) = f(1) - f(0), \quad c_2 = \Delta^2 f(0) = \Delta f(1) - \Delta f(0), \cdots$$

Derangement-like sequences

- An arrangement on $[n]$ is a choice of subset $S \subset [n]$ and a permutation on S.
- An r-cyclic derangement is an "r-signed permutation on $[n]$" with no fixed points. (Formally: action of $C_r \wr S_n$ on $[r] \times [n]$.)
- A cycle-restricted permutation is a permutation whose cycles lengths are in a pre-chosen set $L \subset \mathbb{N}$. Derangements are obtained from $L = \{2, 3, 4, \ldots\}$.

n	0	1	2	3	4	5	6	7	8	9	10
$d^L(n)$	1	1	1	3	9	21	81	351	1233	46089	434241

Table 2. Number of cycle restricted permutations, $L = \{1, 3, 9, 27, 81, \ldots\}$.

Theorem (O'Desky–R, 2022)

Let $d^L(n)$ denote the number of cycle-restricted derangements with respect to L.

1. If $1 \in L$, then $n \mapsto d^L(n)$ is p-adic continuous if and only if $p \in L$.
2. If $1 \notin L$, then $n \mapsto (-1)^n d^L(n)$ is p-adic continuous if and only if $p \notin L$.

Counting formulas:

arrangements $\quad a(n) = \sum_{k=0}^{n} \binom{n}{k} k!$

r-cyclic derangements $\quad (-1)^n d(r, n) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k r^k k!$

L cycle restricted permutations $\quad \sum_{n \geq 0} d^L(n) \frac{X^n}{n!} = \prod_{\ell \in L} \exp\left(\frac{X^\ell}{\ell}\right)$

The last formula is derived by theory of combinatorial species.

Incomplete gamma function

The incomplete gamma function $\Gamma(s, z)$ is defined by

$$\Gamma(s, z) = \int_z^\infty t^s e^{-t} \frac{dt}{t}$$

Theorem (O'Desky-R, '22)

There exists a p-adic continuous $\Gamma_p : \mathbb{Z}_p \times (1 + p\mathbb{Z}_p) \to \mathbb{Z}_p$ such that

$$\Gamma_p(n, r) = \Gamma(n, r)$$

where Γ is the incomplete gamma function.

Key observation: incomplete gamma values count r-cyclic derangements

$$\Gamma(n+1, 1/r) = e^{-1/r} r^{-n} d(n, r)$$

Further questions

- Factorial $n!$ is not p-adic continuous. Morita defined a p-adic gamma function by

$$\Gamma_p^{\text{Mor}}(n+1) = (-1)^{n+1} \prod_{\substack{1 \leq k \leq n \\ p \nmid k}} k$$

Problem: How is Γ_p^{Mor} related to our p-adic incomplete gamma function?

- Euler derived an evaluation of the divergent sum

$$d(-1) = -(0! + 1! + 2! + 3! + \cdots) \approx 0.697.$$

Problem: Is there a combinatorial interpretation of this constant?

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