# Minors of tree distance matrices

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## Distance matrices and determinants

Distance = number of edges in shortest path between vertices

- Examples of distance matrices:

$$D = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 & 2 \\ 1 & 2 & 0 & 2 & 2 \\ 1 & 2 & 2 & 0 & 2 \\ 1 & 2 & 2 & 2 & 0 \end{pmatrix}$$

Problem: What is the determinant of the distance matrix of a tree?

- For the two examples, $\det D = 32$.

The fact that both matrices have the same determinant is not a coincidence!

### Theorem (Graham–Pollack, 1971)

For any tree on $n$ vertices, $\det D = (-1)^{n-1} 2^{n-2}(n-1)$.

## Submatrices

- Example: Tree shown in Figure 1, $S$ = subset of leaf vertices.

$$D = \begin{pmatrix} 0 & 1 & 2 & 1 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 & 2 & 1 & 2 \\ 2 & 1 & 0 & 3 & 3 & 2 & 1 \\ 1 & 2 & 3 & 0 & 2 & 3 & 4 \\ 1 & 2 & 3 & 2 & 0 & 3 & 4 \\ 2 & 1 & 2 & 3 & 3 & 0 & 3 \\ 3 & 2 & 1 & 4 & 4 & 3 & 0 \end{pmatrix}, \qquad D[S] = \begin{pmatrix} 0 & 2 & 3 & 4 \\ 2 & 0 & 3 & 4 \\ 3 & 3 & 0 & 3 \\ 4 & 4 & 3 & 0 \end{pmatrix},$$

$\det D = 192$ $\qquad$ $\det D[S] = -252$

## Application: Phylogenetics

A phylogenetic tree describes the evolutionary history connecting living organisms

- Problem: What is the most likely phylogenetic tree, given observed biological data?

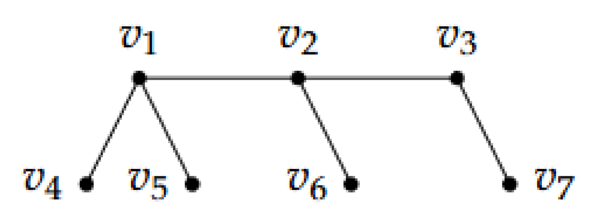


Figure 1. Example tree with 4 leaves.

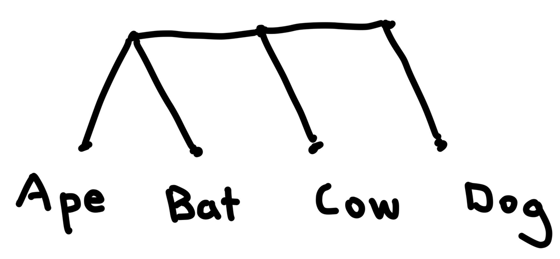


Figure 2. Possible phylogenetic tree.

## Potential theory

Problem: How do particles "distribute" within a region, given repulsive potential?

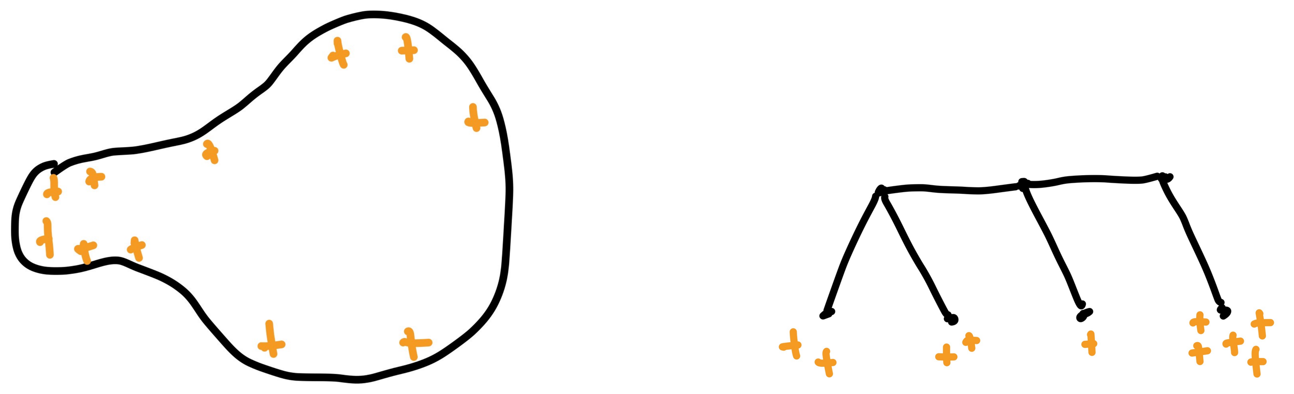

Figure 3. Charged particles on a 2D surface, and on a 1D space.

- Consider "energy" $\mathcal{E}(\mathbf{u}) = -\frac{1}{2}\mathbf{u}^\intercal D[S]\mathbf{u}$
- Equilibrium reached when energy minimized
- Conservation of mass: constrained to $\mathbf{u}$ with $\mathbf{1}^\intercal\mathbf{u} = 1$

Key Observations: (cf. facts on signature of $D[S]$, due to Bapat)

1. $\min\{\mathcal{E}(\mathbf{u}) : \mathbf{1}^\intercal\mathbf{u} = 1\} = -\frac{1}{2}\frac{\det D[S]}{\operatorname{cof} D[S]}$,
2. Minimum $\mathcal{E}(\mathbf{u})$ occurs at $D[S]\mathbf{u}^* = \lambda\mathbf{1}$

Here $\operatorname{cof} A$ is the sum of cofactors, i.e. $\mathbf{1}^\intercal A^{-1}\mathbf{1} = \operatorname{cof} A / \det A$

Consequence of potential theory perspective: monotonicity property

### Theorem (R–Shokrieh-Wu, 2025+)

For any tree $G$, if $A \subset B \subset V(G)$, then

$$\frac{\det D[A]}{\operatorname{cof} D[A]} \leq \frac{\det D[B]}{\operatorname{cof} D[B]}.$$

## Rooted spanning forests

Equilibrium vector $\mathbf{u}^*$ can be expressed in terms of combinatorial quantities

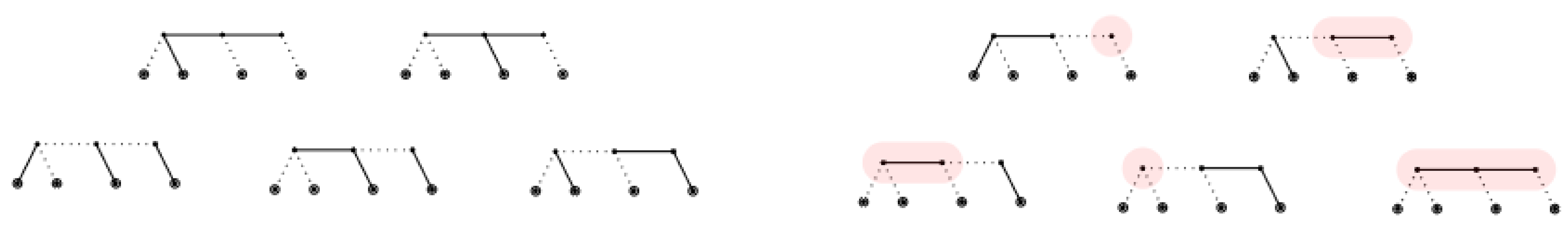

Figure 4. Some $S$-rooted and $(S,*)$-rooted spanning forests

- $\mathcal{F}_1(G;S)$ = $S$-rooted spanning forests of $G$
- $\mathcal{F}_2(G;S)$ = $(S,*)$-rooted spanning forests
- $\kappa_1(G;S)$ and $\kappa_2(G;S)$ = number of respective spanning forests

### Theorem (Bapat–Sivasubramanian, 2011)

"Equilibrium" vector $\mathbf{u}^*$ satisfying $D[S]\mathbf{u}^* = \lambda\mathbf{1}$ and $\mathbf{1}^\intercal\mathbf{u}^* = 1$ is

$$\mathbf{u}_i^* = \frac{1}{2\,\kappa_1(G;S)} \sum_{F\in\mathcal{F}_1(G;S)} (2 - \deg^o(F, v_i)).$$

Bapat–Sivasubramanian used above to prove a combinatorial identity for $\operatorname{cof} D[S]$

- Why does $2 - \deg$ appear? Identity with Laplacian matrix $L$:

$$(LD)_{i,j} = \begin{cases} 2 - \deg(v_i) & \text{if } i \neq j \\ -\deg(v_i) & \text{if } i = j \end{cases}$$

### Theorem (R–Shokrieh–Wu, 2025+)

For any tree $G$ and any vertex subset $S \subset V(G)$, we have

$$\det D[S] = (-1)^{|S|-1} 2^{|S|-2} \left( (n-1)\kappa_1(G;S) - \sum_{\mathcal{F}_2(G;S)} (\deg^o(F,*) - 2)^2 \right).$$

By above and an identity for $\operatorname{cof} D[S]$ by Bapat–Sivasubramanian (2011),

$$\frac{\det D[S]}{\operatorname{cof} D[S]} = \frac{1}{2}\left( (n-1) - \frac{\sum_{\mathcal{F}_2(G;S)} (\deg^o(F,*) - 2)^2}{\kappa_1(G;S)} \right).$$

## Symanzik polynomials

Symanzik polynomials of the first and second kind, $\psi_G$ and $\phi_G$, appear in quantum field theory. For a tree with edge lengths $\{\alpha_e : e \in E(G)\}$,

$$\frac{\det D[S]}{\operatorname{cof} D[S]} = \frac{1}{2}\left( \sum_{E(G)} \alpha_e - \frac{\phi_{(G/S)}(p_{can}; \underline{\alpha})}{\psi_{(G/S)}(\underline{\alpha})} \right), \qquad \text{where } p_{can}(v) = \deg(v) - 2.$$

## Further questions

- $q$-distance analogues,
  $$[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$$
  $q$-analogue of Graham–Pollak found by Bapat–Lal–Pati and Yan–Yeh
- $k$-Steiner distances: for a choice of $k$ vertices, count number of edges in subtree spanned by chosen vertices. Explored by Cooper–Du and Azimi–Sivasubramanian
- "Combinatorial" proof of main identity? Gutierrez–Lillo

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