Tropical weights of Weierstrass points

Omid Amini, Lucas Gierczak *Ecole Polytechnique*, Harry Richman *University of Washington*

Motivation: Weierstrass points on algebraic curves

Weierstrass points are a tool for better understanding algebraic curves.

- If X is a smooth, projective, genus g curve, $x \in X$ is a Weierstrass point if
$$K \sim gx + E \qquad \text{for some effective divisor } E$$
where K denotes the canonical divisor.

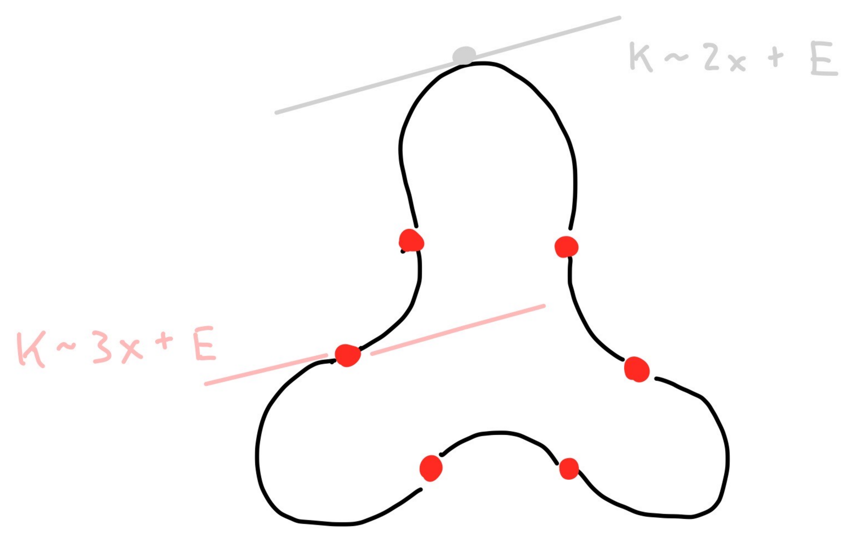

Figure 1. Weierstrass points on a genus 3 algebraic curve.

- Geometric view: the curve X has a canonical map to projective space $\mathbb{P}^{(g-1)}$. Weierstrass points are the inflection points of the canonical embedding.
- The weight of a Weierstrass point is the degree of vanishing of the Wronskian; it records collisions of Weierstrass points on "generic nearby" curves.

Theorem (Hurwitz, 1893)

If X is a smooth, projective algebraic curve of genus $g \geq 2$, then the total weight of Weierstrass points of X is $g^3 - g$.

Motivating Problems

1. Suppose X is a stable nodal curve of genus $g \geq 2$. How many (limit) Weierstrass points does X have? How are they distributed among components of X?
2. Suppose X is a smooth curve over $\mathbb{Z}_p$, and let $X(\mathbb{F}_p)$ denote its reduction mod p. Where are the reductions of Weierstrass points on $X(\mathbb{F}_p)$?

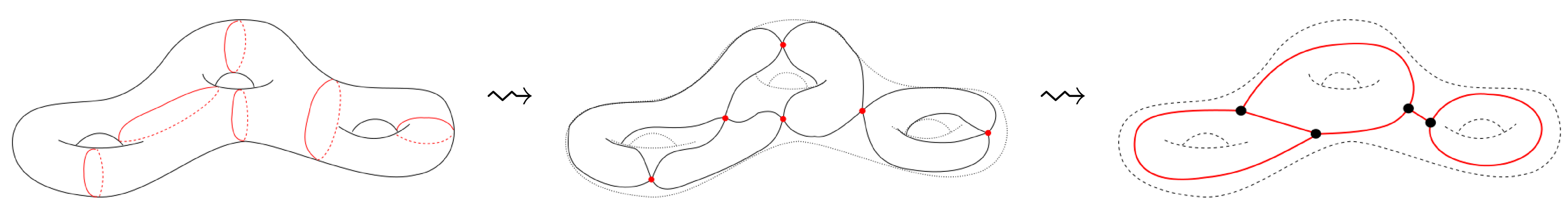

Figure 2. Deforming a smooth curve to a nodal curve, and its dual graph.

Tropical curves

- In a metric graph every edge has a real, positive length. A deformation of a smooth curve to a nodal curve can be encoded in a metric graph.
- We call a metric graph a tropical curve to emphasize an analogy with algebraic curves; this analogy carries over to divisors and linear equivalence.
- The canonical divisor of a tropical curve is the divisor $K = \sum_{x \in \Gamma} (\mathrm{val}(x) - 2)x$

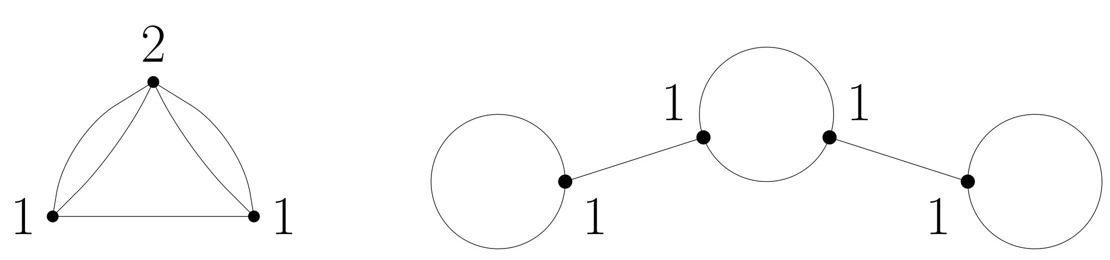

Figure 3. Canonical divisors of tropical curves.

- If Γ is a tropical curve of genus g, $x \in \Gamma$ is a Weierstrass point if
$$K \sim gx + E \qquad \text{for some effective divisor } E.$$
- The Weierstrass locus $L_W = L_W(\Gamma)$ is the subset of all Weierstrass points.

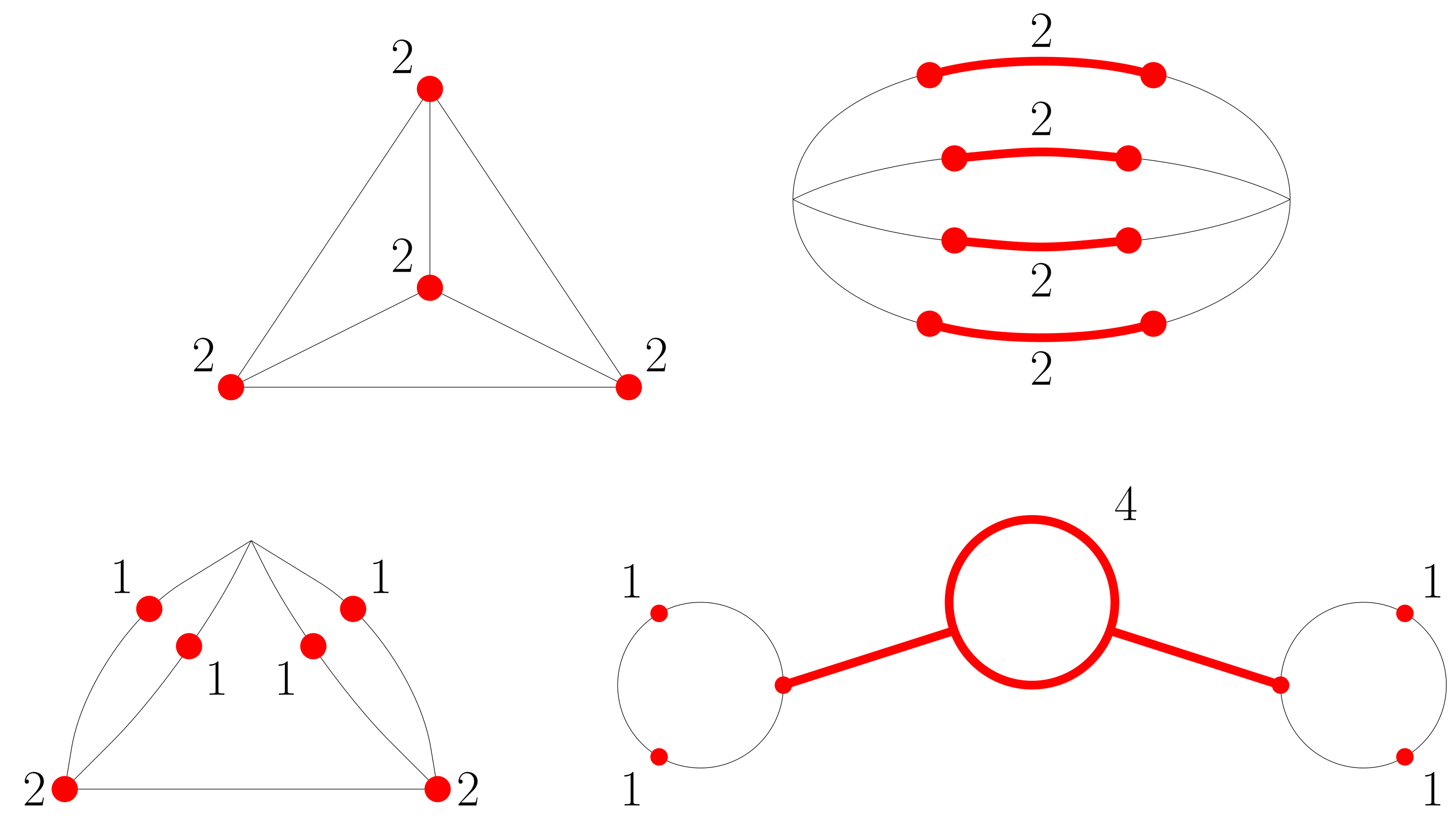

Figure 4. Weierstrass locus and weights of tropical curves.

Tropical Problem

What is a "good" definition of weights for tropical Weierstrass points?

- Components of L_W can be isolated points, or can have positive length.
- We call a subset A of a tropical curve L_W-measurable if $A \cap L_W$ contains only "whole components" of L_W.

Tropical Weierstrass weights

For a closed, connected subset $A \subset \Gamma$, the Weierstrass weight of A is
$$w(A) = (g+1)(g(A) - 1) - \sum_{\nu \in \partial A} (s_0^\nu(K) - 1),$$
where

- $g(A)$ is the genus, i.e. first Betti number, of A;
- ∂A is the set of outgoing directions from A;
- $s_0^\nu(K)$ is the minimum slope along ν in $\mathrm{Rat}(K)$.

In particular, the total Weierstrass weight $w(\Gamma)$ is $g^2 - 1$.

Theorem (AGR, 2023)

1. The Weierstrass weight $w(A)$ is additive on L_W-measurable subsets of Γ.
2. If A is L_W-measurable, then for any algebraic curve X tropicalizing to Γ, the total weight of Weierstrass points tropicalizing to A is $g \cdot w(A)$.

Slope sets

- For each oriented segment ν of Γ, denote the minimum slope
$$s_0^\nu(K) = \min\{\mathrm{slope}_\nu(f) : f \in \mathrm{Rat}(K)\}$$
where $\mathrm{Rat}(K)$ = all piecewise-linear functions on Γ whose poles are bounded by K.
- Minimum slopes are attained for reduced divisors.

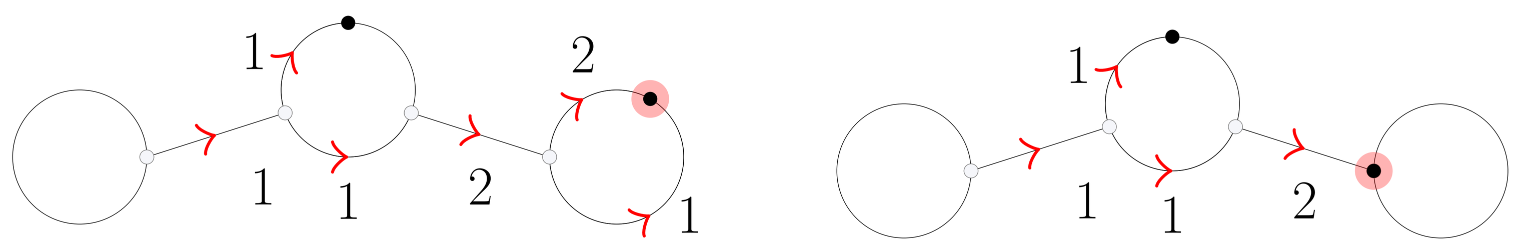

Figure 5. Minimum slopes $\{s_0^\nu(K)\} = \{-1, -2\}$ and $\{s_0^\nu(K)\} = \{0, 0, -2\}$, via reduced divisor.

- Calculate Weierstrass weights via minimum slopes:

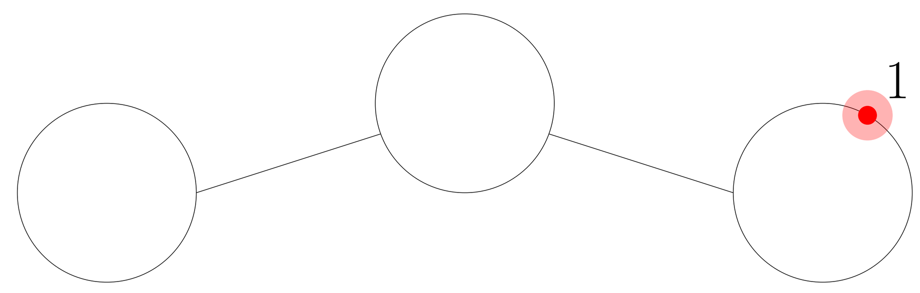

Figure 6. Weierstrass weight $w(A) = (3+1)(0-1) - (-2-3) = 1$.

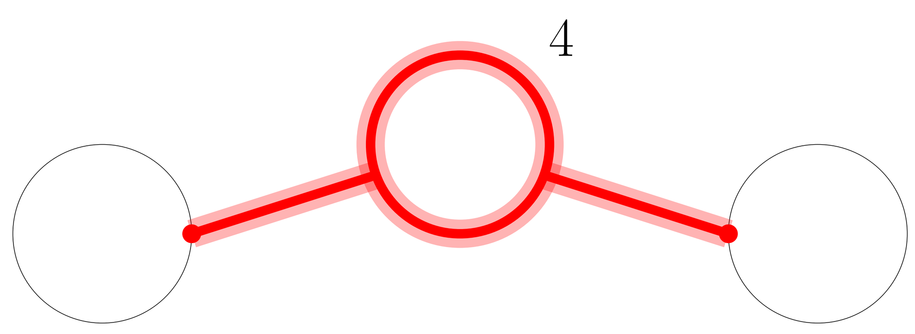

Figure 7. Weierstrass weight $w(A) = (3+1)(1-1) - (-1-1-1-1) = 4$.