Uniform bounds on tropical torsion points

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Tropical curves and Jacobians

A metric graph is a graph where every edge is assigned a real, positive length. Given a graph G, the choice of edge lengths is parametrized by a point in $\mathbb{R}_{>0}^{|E(G)|}$.

We call a metric graph a tropical curve to emphasize an analogy with Riemann surfaces, which are complex algebraic curves. This analogy carries over the construction of divisors, linear equivalence, and Jacobians.

Torsion points

If Γ is a tropical curve of genus g, the Jaocbian of Γ is a real torus of dimension g, $\mathrm{Jac}(\Gamma) = \mathbb{R}^g/\mathbb{Z}^g$. The torsion subgroup of Γ is $\mathrm{Jac}(\Gamma)_{\mathrm{tors}} = \mathbb{Q}^g/\mathbb{Z}^g$.

A torsion point on a pointed tropical curve (Γ, q) is a point x such that

$$[nx] \sim [nq] \qquad \text{for some positive integer } n.$$

Equivalently, if $\iota_q(\Gamma) : \Gamma \to \mathrm{Jac}(\Gamma)$ is the Abel–Jacobi embedding of a curve into its Jacobian, then the torsion points are the points in the intersection

$$\iota_q(\Gamma) \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}}.$$

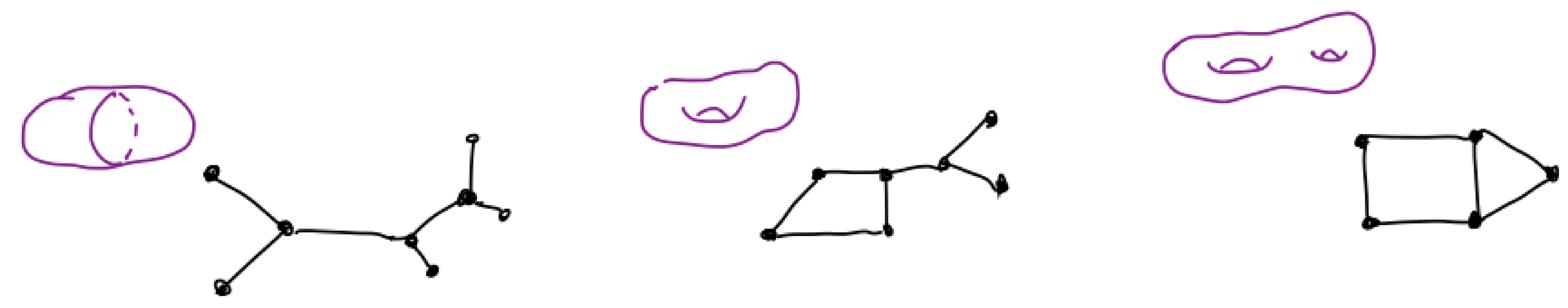

Figure 1. Some tropical curves of genus 0, 1, and 2.

Theorem (R, 2022)

1. For a metric graph Γ of genus g, if the number of torsion points is finite then
$$\#(\iota_x(\Gamma) \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}}) \leq 3g - 3.$$
2. If G is a biconnected graph of genus $g \geq 2$, then $\Gamma = (G, \ell)$ has finitely many torsion points for very general edge lengths ℓ.

We say a subset of $\mathbb{R}^n$ is very general if its complement is a countable union of positive-codimension algebraic subsets. A (metric) graph is biconnected if any two points can be connected by two vertex-disjoint paths.

Examples

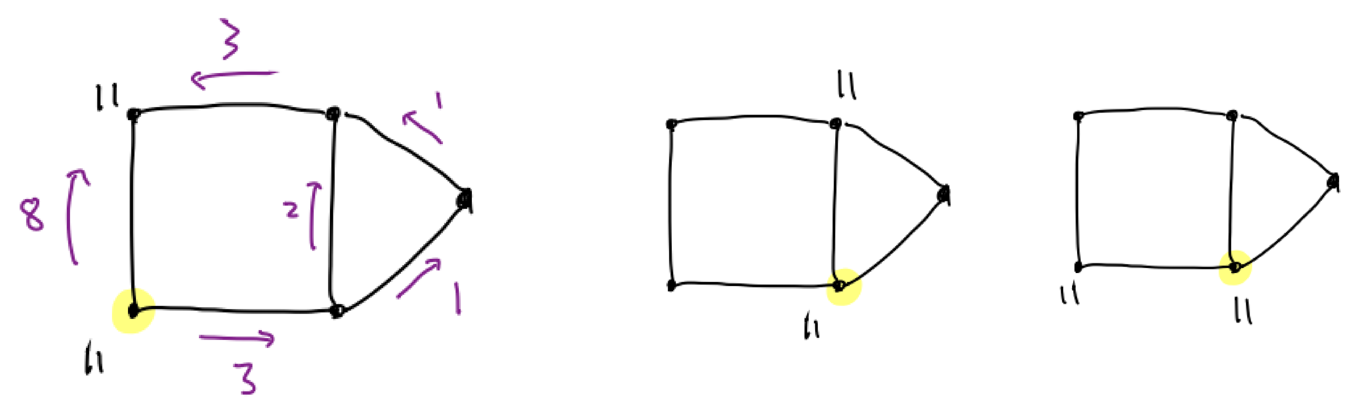

Figure 2. Tropical curve with unit edge lengths.

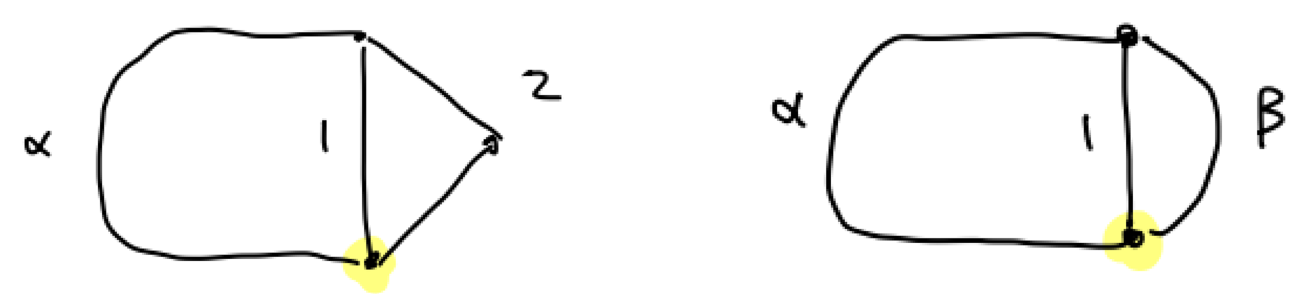

Figure 3. Tropical curves with some irrational lengths.

Some pointed tropical curves (Γ, q) with basepoint q highlighted. Assume α and β are irrational numbers. Which have infinitely many torsion points?

Higher-degree

The degree d Abel–Jacobi map

$$\iota_q^{(d)} : \Gamma^d \to \mathrm{Jac}(\Gamma)$$

sends a degree d effective divisor D to the linear equivalence class $[D - dq]$.

For arbitrary $d \geq 1$, we may ask: how large is the intersection of the image $\iota_q^{(d)}(\Gamma^d)$ with $\mathrm{Jac}(\Gamma)_{\mathrm{tors}}$?

Theorem (R, 2022)

1. For a metric graph Γ of genus g, if the number of d-torsion points is finite then
$$\#(\iota_q^{(d)}(\Gamma^d) \cap \mathrm{Jac}(\Gamma)_{\mathrm{tors}}) \leq \binom{3g-3}{d}.$$
2. If G is a biconnected graph of genus $g \geq 2$, then $\Gamma = (G, \ell)$ has finitely many d-torsion points for very general edge lengths if and only if $d < \gamma^{\mathrm{ind}}(\Gamma)$, the independent girth.

Independent girth

The independent girth of a graph G is the minimal cographic rank of a cycle:

$$\gamma^{\mathrm{ind}}(\Gamma) = \min_{C \in \mathcal{C}(G)} \mathrm{rk}^{\perp}(C) := \min_{C \in \mathcal{C}(G)} |E(C)| + 1 - h_0(G \setminus C).$$

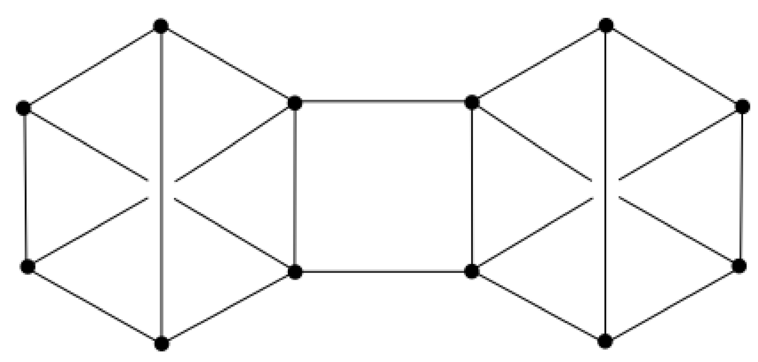

Figure 4. Graph with independent girth $\gamma^{\mathrm{ind}} = 3$.

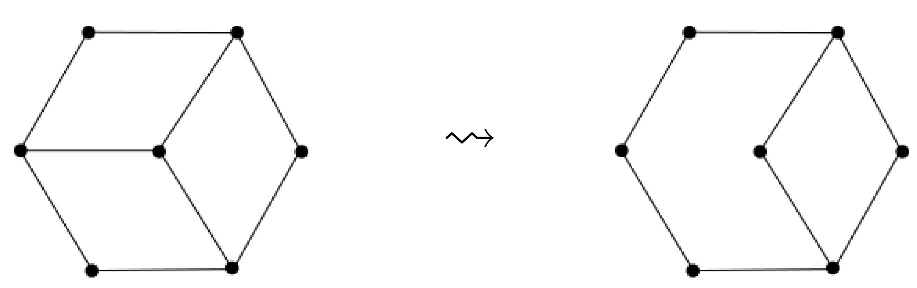

Figure 5. Independent girth may decrease under edge deletion.

Motivation: Rational points

Mordell conjectured in 1922 that a smooth algebraic curve of genus $g \geq 2$ has only a finite number of rational points. Later, Manin and Mumford made a similar conjecture for the number of torsion points. It is suspected that these bounds cans be made uniform in the genus g. In the case of torsion points, the uniform conjecture was recently proved by Kühne and Looper–Silverman–Wilmes.

Theorem (Faltings, 1983)

Suppose X is a smooth algebraic curve of genus $g \geq 2$. Then $\#X(\mathbb{Q}) < \infty$.

Conjecture (Uniform Faltings)

Suppose X is a smooth algebraic curve of genus $g \geq 2$. There is a function $N(g)$ such that $\#X(\mathbb{Q}) < N(g)$.

Theorem (Raynaud, 1983)

Suppose X is a smooth algebraic curve of genus $g \geq 2$. Then $\#(\iota_q(X) \cap \mathrm{Jac}(X)_{\mathrm{tors}}) < \infty$.

Conjecture (Uniform Raynaud)

Suppose X is an algebraic curve of genus $g \geq 2$. There is a function $M(g)$ such that $\#(\iota_q(X) \cap \mathrm{Jac}(X)_{\mathrm{tors}}) < M(g)$.

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