ON DETERMINANTS OF RESISTANCE MATRICES

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Abstract. We prove a new combinatorial identity for the determinant of the resistance matrix of a finite graph, which involves counts of spanning trees and forests. This generalizes a result of Graham and Pollak on distances matrices of trees. We make use of Bapat's expression of the resistance matrix determinant as a linear algebraic quantity.

1. Introduction

Suppose $G = (V, E)$ is a finite connected graph. We allow G to have parallel edges, but not loops. The *resistance matrix* of G is defined as the matrix in $\mathbb{R}^{|V|\times|V|}$ having entries

$$R_{ij} = \text{effective resistance between } v_i \text{ and } v_j \text{ in } G,$$

when G is considered as an electrical network with a unit resistor on each edge. Combinatorially, this quantity is equal to a ratio of spanning tree counts [10],

$$R_{ij} = \frac{\kappa(G/ij)}{\kappa(G)},$$

where $\kappa(G)$ denotes the number of spanning trees and G/ij denotes the graph obtained by identifying (or "gluing together") vertices v_i and v_j.

In this paper, we provide a combinatorial formula for the determinant of the resistance matrix. Given a finite graph G, let $\kappa_2(G)$ denote the number of two-component spanning forests.

Theorem 1. *Let $G = (V, E)$ be a graph with n vertices and resistance matrix R. Then*

$$\det R = \frac{(-1)^{n-1}2^n}{\kappa(G)} \left(\frac{1}{3}\frac{\kappa_2(G)}{\kappa(G)} - \frac{1}{12}\sum_{e\in E} \frac{\kappa(G/e)^2}{\kappa(G)^2} \right) \tag{1}$$

where G/e is the edge contraction of G at e.

This theorem is proved by combining two existing results in the literature: one by Bapat [2] on the determinant of the resistance matrix, and one by the current authors [13] on identities involving counts of spanning forests and resistances. For some examples of resistance matrices and their determinants, see §4.

We also give a second alternative expression for the determinant of the resistance matrix that is less explicitly combinatorial.

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Theorem 2. *Let R be the resistance matrix of a graph $G = (V, E)$ with n vertices. Then for any vertex index $k \in \{1, \ldots, n\}$,*

$$\det R = \frac{(-1)^{n-1} 2^{n-2}}{\kappa(G)} \sum_{\substack{e \in E \\ e=(i,j)}} (R_{ik} - R_{jk})^2 . \tag{2}$$

Here we use "$e = (i, j)$" to mean that e connects vertices v_i and v_j, by abuse of notation. This result states that, surprisingly, the computation of $\det R$ may be reduced to an expression using only entries of any *single column* of R, up to the term $\kappa(G)$.

For a general graph G, the off-diagonal entries of the resistance matrix are fractions with denominator $\kappa(G)$. Thus, we would expect a priori that $\det R$ has $\kappa(G)^n$ as its denominator. Our theorems show that, to the contrary, the denominator of $\det R$ is at most $\kappa(G)^3$. This property of the denominator of $\det R$ also extends to weighted graphs, if $\kappa(G)$ is replaced with the spanning tree polynomial $\kappa(G, \alpha)$; see §3.1. This unexpected cancellation was observed experimentally by Faber [8, Remark 4.7], in connection to arithmetic geometry.

From a computational perspective, the expression (2) is simpler than (1) because it avoids the $\kappa_2(G)$ term—in fact, an efficient way to compute the number of two-forests κ_2 for a general graph, over a brute-force search, is to take advantage of the equality between (1) and (2).

Remark 3. If G is a tree, the effective resistance coincides with shortest-path distance. Graham and Pollak [9] proved that, in this case,

$$\det R = (-1)^{n-1} 2^{n-2} (n - 1). \tag{3}$$

This identity shows that the determinant depends on surprisingly little information of the underlying tree. Their result led to a large amount of subsequent investigations on tree distance matrices and their generalizations.

We can obtain (3) from Theorem 1 since, for a tree, we have $\kappa(G) = 1$ and

$$\frac{1}{3}\kappa_2(G) - \frac{1}{12} \sum_{e \in E(G)} \kappa(G/e)^2 = \frac{1}{3}(n-1) - \frac{1}{12}(n-1).$$

It is also clear that (3) follows from Theorem 2, since for a tree $(R_{ik} - R_{jk})^2 = 1$ given any edge $e = (i, j)$.

Remark 4. In other recent work [11], we generalize the Graham–Pollak tree identity (3) to identities for any principal minor of the distance matrix.

There is also an identity for the principal minors of the resistance matrix, in the ongoing work [12]. This result is more technical to state, and is motivated by questions from arithmetic geometry [6].

Remark 5. One particular generalization of (3), found by Bapat–Kirkland–Neumann [1], is the following identity for the distance matrix of an edge-weighted graph. If the distance matrix D takes into account postive real weights on edges $\{\alpha_e : e \in E\}$, then

$$\det D = (-1)^{n-1} 2^{n-2} \prod_{e \in E} \alpha_e \sum_{e \in E} \alpha_e. \tag{4}$$

There are analogous weighted versions of Theorem 1 and 2, see §3.1.

1.1. **Organization.** In §2 we review facts about effective resistance and the resistance curvature of a graph. In §3 we prove our main results. In §4 we provide some examples of resistance matrices of specific graphs.

2. Effective resistance and graph curvature

When working with a graph $G = (V, E)$, we implicitly assume that the vertices are ordered, $V = \{v_1, \ldots, v_n\}$, and that the edges come equipped with a chosen orientation, and we write e^+, e^- for the endpoints of the edge e. The choice of order and orientation is arbitrary, and will not have any effect on the results.

For notation, we use $r(v_i, v_j)$ to denote the effective resistance between vertices v_i and v_j. The *resistance matrix* is the matrix R in $\mathbb{R}^{|V|\times|V|}$ with entries $R_{ij} = r(v_i, v_j)$. Unless noted otherwise, when discussing effective resistance we always assume that each edge has unit resistance. As mentioned in the introduction,

$$r(v_i, v_j) = \frac{\kappa(G/ij)}{\kappa(G)},$$

where G/ij denotes the graph obtained from gluing together vertices v_i and v_j. For a reference, see Kirchhoff [10] or Biggs [5, Section 17] for a more modern treatment. For an edge e, the effective resistance between its endpoints is

$$r(e^+, e^-) = \frac{\kappa(G/e)}{\kappa(G)} \tag{5}$$

where G/e is the edge contraction; see e.g. [5, Proposition 17.1].

2.1. **Graph curvature.** Here we recall the definition of a vector in $\mathbb{R}^{|V|}$ that has a special relation to the resistance matrix. The following terminology is due to Devriendt–Lambiotte [7, Definition 1, p. 5]. The same vector, up to a scaling, appeared earlier in Bapat [2, p. 76].

Definition 6. The *resistance curvature* of a graph $G = (V, E)$ is the vector $\boldsymbol{\mu} \in \mathbb{R}^{|V|}$ with components

$$\boldsymbol{\mu}_i = 1 - \frac{1}{2} \sum_{e \in N(v_i)} r(e^+, e^-), \tag{6}$$

where $N(v_i)$ denotes the set of edges incident to vertex v_i.

We will often abbreviate "resistance curvature" to just "curvature" of G.

Proposition 7. *The curvature $\boldsymbol{\mu}$ of a graph G is the unique vector $\boldsymbol{\mu} \in \mathbb{R}^{|V|}$ that satisfies the two conditions*

(a) $R\boldsymbol{\mu} = \lambda \mathbf{1}$ for some real number λ;
(b) $\mathbf{1}^\intercal \boldsymbol{\mu} = 1$, i.e. the entries of $\boldsymbol{\mu}$ sum to one.

Proof. Let $\boldsymbol{\tau} = 2\boldsymbol{\mu}$. In [2, Equation (10)] Bapat shows that $R\boldsymbol{\tau} = \dfrac{\boldsymbol{\tau}^\intercal R \boldsymbol{\tau}}{2}\mathbf{1}$, which is equivalent to $R\boldsymbol{\mu} = (\boldsymbol{\mu}^\intercal R \boldsymbol{\mu})\mathbf{1}$. In [2, p. 77] it is shown that $\mathbf{1}^\intercal \boldsymbol{\tau} = 2$.

Alternatively, see Devriendt–Lambiotte [7, Equation (22), p. 19]. □

2.2. **Determinant expressions.**

Theorem 8 (Bapat [2, Theorem 4])**.** *If R is the resistance matrix of a graph, and $\boldsymbol{\mu}$ is the curvature, then* $\det R = \dfrac{(-2)^{n-1}}{\kappa(G)}\boldsymbol{\mu}^\intercal R\boldsymbol{\mu}$.

In Bapat's notation, this result in stated in terms of $\boldsymbol{\tau} = 2\boldsymbol{\mu}$, with the identity

$$\det R = \frac{(-2)^{n-3}}{\kappa(G)}\boldsymbol{\tau}^\intercal R\boldsymbol{\tau}.$$

Bapat proves Theorem 8 by first finding an expression for the inverse matrix R^{-1}, and then computing the determinant of R^{-1}. Here, for the convenience of the reader, we sketch an alternative argument, along the lines used in [11].

Proof of Theorem 8. First, note that

$$\frac{\det R}{\operatorname{cof} R} = \frac{1}{\mathbf{1}^\intercal(R^{-1})\mathbf{1}},$$

where $\operatorname{cof} A$ denotes the *sum of cofactors* $\sum_{i,j}(-1)^{i+j}\det A_{i,j}$. Then, we may argue that

$$\frac{1}{\mathbf{1}^\intercal(R^{-1})\mathbf{1}} = \max\{\mathbf{u}^\intercal R\mathbf{u} : \mathbf{u} \in \mathbb{R}^V,\ \mathbf{1}^\intercal\mathbf{u} = 1\}$$

by using the method of Lagrange multipliers and the fact that R has signature $(1, n-1)$.[1] Furthermore, the maximum value of $\mathbf{u} \mapsto \mathbf{u}^\intercal R\mathbf{u}$ is achieved precisely at the curvature vector $\boldsymbol{\mu}$ due to Proposition 7. It then follows that

$$\frac{\det R}{\operatorname{cof} R} = \boldsymbol{\mu}^\intercal R\boldsymbol{\mu}.$$

Finally, we use the sum-of-cofactors identity $\operatorname{cof} R = (-2)^{n-1}/\kappa(G)$ which is a special case of [3, Theorem 7]. □

3. Proofs

We now prove our main theorem.

Proof of Theorem 1. A result of Bapat [2, Theorem 4] states that

$$\det R = \frac{(-2)^{n-1}}{\kappa(G)}\boldsymbol{\mu}^\intercal R\boldsymbol{\mu}, \tag{7}$$

where $\boldsymbol{\mu}$ is the curvature vector (see §2.1). By [13, Proposition 5.3], we have the identity

$$\boldsymbol{\mu}^\intercal R\boldsymbol{\mu} = 2\gamma(G) \ :=\ \max\{\mathbf{u}^\intercal R\mathbf{u} : \mathbf{u} \in \mathbb{R}^V,\ \mathbf{1}^\intercal\mathbf{u} = 1\}. \tag{8}$$

Finally, using [13, Theorem A] we have

$$\gamma(G) = \frac{1}{3}\frac{\kappa_2(G)}{\kappa(G)} - \frac{1}{12}\sum_{e\in E} r(e^+, e^-)^2. \tag{9}$$

Together, (7), (8), and (9) show that

$$\det R = \frac{(-1)^{n-1}2^n}{\kappa(G)}\left(\frac{1}{3}\frac{\kappa_2(G)}{\kappa(G)} - \frac{1}{12}\sum_{e\in E} r(e^+, e^-)^2\right).$$

[1] We can avoid reference to the signature of R by replacing "maximum" with "critical value".

This proves the desired result, after making use of the resistance identity (5) for $r(e^+, e^-)$. □

Proof of Theorem 2. We follow the same steps as above, except that in place of (9) we instead use [13, Proposition 5.4 (c)]

$$\gamma(G) = \frac{1}{4}\sum_{e\in E}(r(e^+,q) - r(e^-,q))^2. \tag{10}$$ □

3.1. **Edge-weighted graphs.** If $G = (V, E)$ is a connected graph equipped with edge resistances $\{\alpha_e : e \in E\}$, then the effective resistance R_{ij} between vertices v_i and v_j satisfies $R_{ij} = \dfrac{\kappa(G/ij, \alpha)}{\kappa(G, \alpha)}$, where

$$\kappa(G,\alpha) = \sum_{T\in\mathcal{T}(G)} \prod_{e\notin T} \alpha_e$$

and $\mathcal{T}(G)$ denotes the set of spanning trees of G, and G/ij denotes the same quotient graph as earlier. Similarly, we let $\kappa_2(G, \alpha)$ denote the weighted sum

$$\kappa_2(G,\alpha) = \sum_{F\in\mathcal{F}_2(G)} \prod_{e\notin F} \alpha_e.$$

over the set of two-component spanning forests $\mathcal{F}_2(G)$ [5, Sections 15-17]. Then, we have the identity

$$\det R = \frac{(-1)^{n-1}2^n\prod_E \alpha_e}{\kappa(G,\alpha)}\left(\frac{1}{3}\frac{\kappa_2(G,\alpha)}{\kappa(G,\alpha)} - \frac{1}{12}\sum_{e\in E}\alpha_e\frac{\kappa(G/e,\alpha)^2}{\kappa(G,\alpha)^2}\right). \tag{11}$$

Alternatively, the weighted version of Theorem 2 is

$$\det R = \frac{(-1)^{n-1}2^{n-2}\prod_E \alpha_e}{\kappa(G,\alpha)}\sum_{\substack{e\in E\\ e=(i,j)}}\frac{(R_{ik}-R_{jk})^2}{\alpha_e}, \tag{12}$$

for any vertex index k. Some examples (Examples 12 and 13) are given in §4.

3.2. **Edge-transitive graphs.** We note that Theorem 1 can be further simplified if G is sufficiently symmetric. Recall that G is *edge-transitive* if, for any $e, e' \in E(G)$, there is some automorphism of G that sends e to e'. For a nice exposition on edge-transitive graphs, and other symmetry classes, see Biggs [4, Chapter 15].

The next result follows from Theorem 1.

Corollary 9. *If G is an edge-transitive graph with n vertices and m edges, then*

$$\det R = \frac{(-1)^{n-1}2^n}{\kappa(G)}\left(\frac{1}{3}\frac{\kappa_2(G)}{\kappa(G)} - \frac{1}{12}\frac{(n-1)^2}{m}\right). \tag{13}$$

Proof. Since G is edge-transitive, $\kappa(G/e) = \kappa(G/e')$ for any two edges $e, e' \in E(G)$. It follows from a straightforward counting argument that in this case, $\dfrac{\kappa(G/e)}{\kappa(G)} = \dfrac{|V|-1}{|E|}$ for every edge e. □

4. Examples

Example 10 (House graph)**.** Consider the house graph, shown in Figure 1. This graph has $\kappa(G) = 11$ spanning trees, and $\kappa_2(G) = 19$ two-component spanning forests.

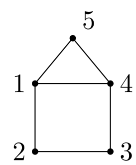

Figure 1. House graph.

Its resistance matrix is

$$R = \frac{1}{11}\begin{pmatrix} 0 & 8 & 10 & 6 & 7 \\ 8 & 0 & 8 & 10 & 13 \\ 10 & 8 & 0 & 8 & 13 \\ 6 & 10 & 8 & 0 & 7 \\ 7 & 13 & 13 & 7 & 0 \end{pmatrix}.$$

A priori, we would expect that $\det R$ has 11^5 as its denominator. Instead, due to cancellation in accordance with Theorem 1 we have

$$\det R = \frac{1360}{11^3}.$$

In terms of the theorem,

$$\det R = \frac{2^5}{11}\left(\frac{1}{3}\cdot\frac{19}{11} - \frac{1}{12}\cdot\frac{6^2+7^2+7^2+8^2+8^2+8^2}{11^2}\right).$$

Applying Theorem 2 with $k = 1$, we have

$$\det R = \frac{2^3}{11}\left(\frac{8^2+6^2+7^2+(10-8)^2+(10-6)^2+(7-6)^2}{11^2}\right).$$

Example 11 (Cube graph)**.** The cube graph has 8 vertices and 12 edges; see Figure 2. This graph has $\kappa(G) = 384$ spanning trees. For the number of two-forests, we have:

$$\begin{aligned}\kappa_2(G) &= \#(\text{all 6-edge subgraphs}) - \#(\text{those containing cycles}) \\ &= \binom{12}{6} - \left(6\cdot\binom{8}{2} + 12 + 4\right) = 740.\end{aligned}$$

Its resistance matrix is

$$R = \frac{1}{12}\begin{pmatrix} 0 & 7 & 9 & 7 & 9 & 10 & 9 & 7 \\ 7 & 0 & 7 & 9 & 10 & 9 & 7 & 9 \\ 9 & 7 & 0 & 7 & 9 & 7 & 9 & 10 \\ 7 & 9 & 7 & 0 & 7 & 9 & 10 & 9 \\ 9 & 10 & 9 & 7 & 0 & 7 & 9 & 7 \\ 10 & 9 & 7 & 9 & 7 & 0 & 7 & 9 \\ 9 & 7 & 9 & 10 & 9 & 7 & 0 & 7 \\ 7 & 9 & 10 & 9 & 7 & 9 & 7 & 0 \end{pmatrix}.$$

We have $\det R = -\dfrac{86\,593\,536}{12^8} = -\dfrac{29}{12^2}$.

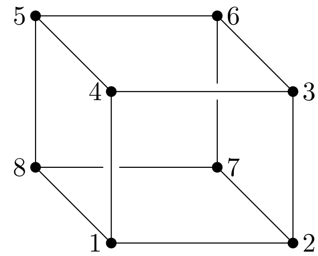

FIGURE 2. Cube graph

Note that the cube graph is edge-transitive. In terms of Corollary 9,

$$\det R = -\frac{2^8}{384}\left(\frac{1}{3}\cdot\frac{740}{384} - \frac{1}{12}\cdot\frac{7^2}{12}\right).$$

In terms of Theorem 2, with $k = 1$, we have

$$\det R = -\frac{2^6}{384}\left(\frac{3\times 7^2 + 6\times 2^2 + 3\times 1^2}{12^2}\right).$$

Example 12 (Triangle graph)**.** Suppose G is the triangle graph with general edge weights a, b, and c. Then the resistance matrix is

$$R = \frac{1}{a+b+c}\begin{pmatrix} 0 & ab+ac & ab+bc \\ ab+ac & 0 & ac+bc \\ ab+bc & ac+bc & 0 \end{pmatrix}.$$

Its determinant is

$$\det R = \frac{2abc(a+b)(a+c)(b+c)}{(a+b+c)^3}.$$

The curvature vector for this graph is

$$\boldsymbol{\mu} = \frac{1}{2(a+b+c)}\begin{pmatrix} a+b \\ a+c \\ b+c \end{pmatrix}.$$

This graph has $\kappa(G;\alpha) = a+b+c$ and $\kappa_2(G;\alpha) = ab+ac+bc$. The weighted version of the main theorem (11) states that we have

$$\det R = 2^3\cdot\frac{abc}{a+b+c}\left(\frac{1}{3}\cdot\frac{ab+ac+bc}{a+b+c} - \frac{1}{12}\cdot\frac{a(b+c)^2+b(a+c)^2+c(a+b)^2}{(a+b+c)^2}\right)$$

Alternatively, if we use (12) with $k = 1$, we have

$$\det R = 2\cdot\frac{abc}{a+b+c}\left(\frac{1}{a}\cdot\frac{(ab+ac)^2}{(a+b+c)^2} + \frac{1}{b}\cdot\frac{(ab+bc)^2}{(a+b+c)^2} + \frac{1}{c}\cdot\frac{(ac-bc)^2}{(a+b+c)^2}\right).$$

Example 13 (Theta graph)**.** Consider the theta graph, i.e. the unique multigraph with two vertices, three edges, and no loops. For the theta graph with general edge weights a, b, and c, the resistance matrix is

$$R = \frac{1}{ab+ac+bc}\begin{pmatrix} 0 & abc \\ abc & 0 \end{pmatrix}.$$

so $\det R = -\left(\frac{abc}{ab+ac+bc}\right)^2$. This graph has $\kappa(G;\alpha) = ab+ac+bc$ and $\kappa_2(G;\alpha) = abc$. The main theorem (11) states that we have

$$\det R = -2^2 \cdot \frac{abc}{ab+ac+bc}\left(\frac{1}{3}\cdot\frac{abc}{ab+ac+bc} - \frac{1}{12}\cdot\frac{ab^2c^2+a^2bc^2+a^2b^2c}{(ab+ac+bc)^2}\right).$$

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