Distinguishing graphs with tropical Weierstrass weights

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IBS Discrete Math Seminar

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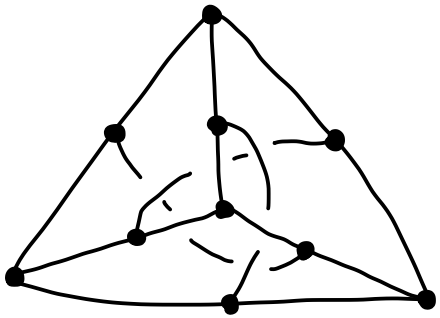
joint w/ Omid Amini (Paris-Saclay) & Lucas Gierczak (Marseille)

Graph Isomorphism problem

Q: Decide whether finite graphs G, H are isomorphic

Ex.



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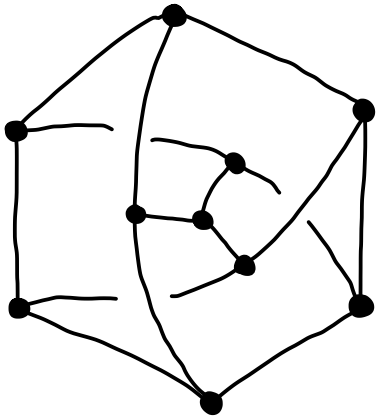

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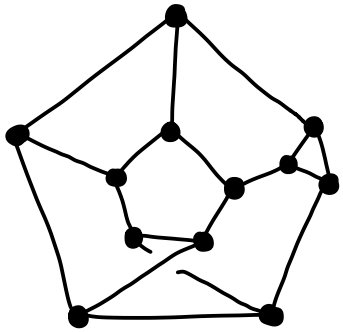

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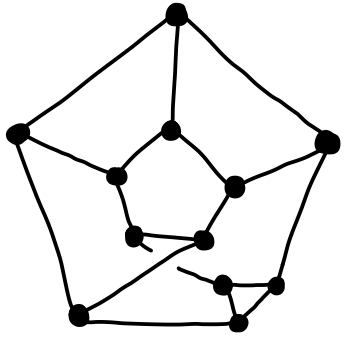

Graph Isomorphism problem

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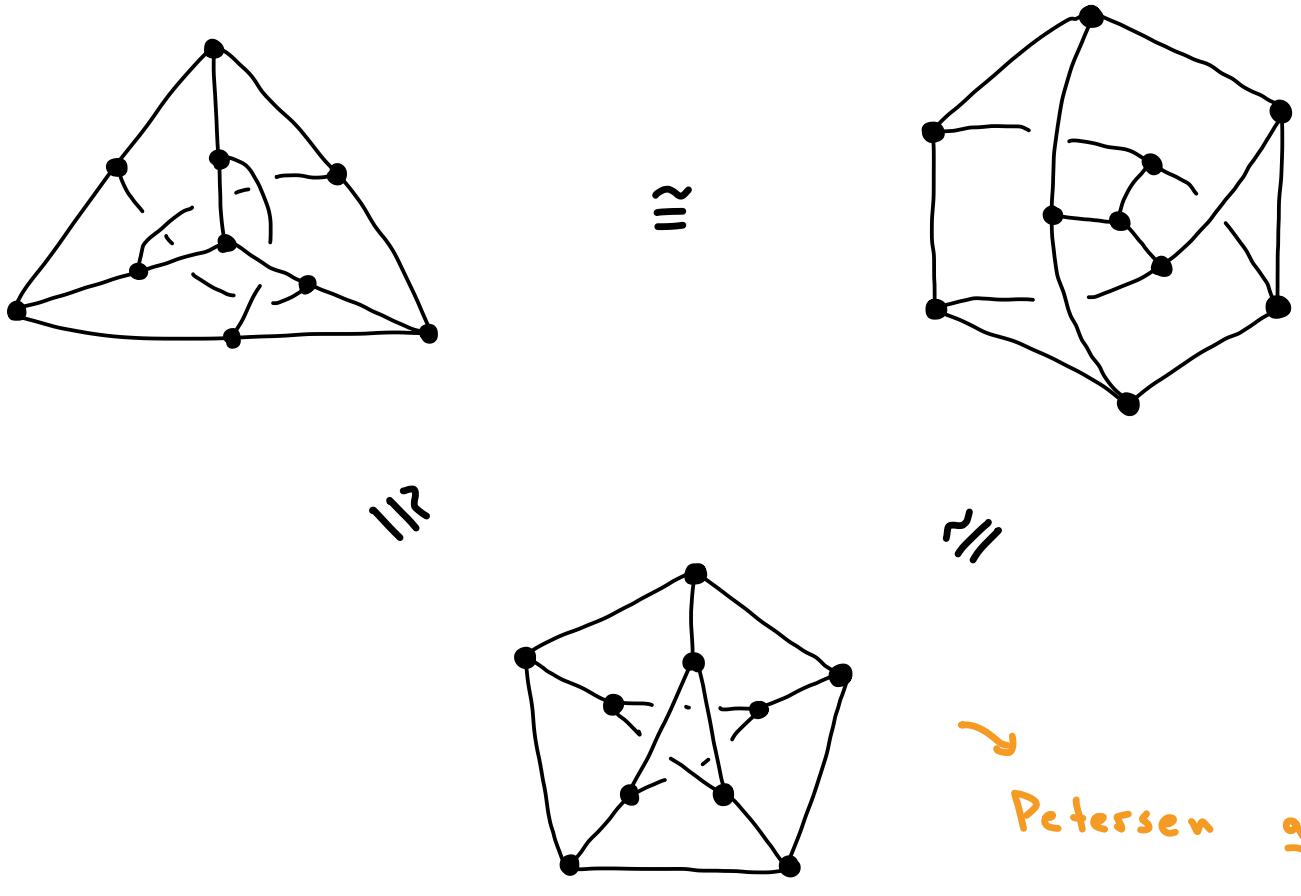


$\cong$

$\not\cong$

$\not\cong$

Petersen graph

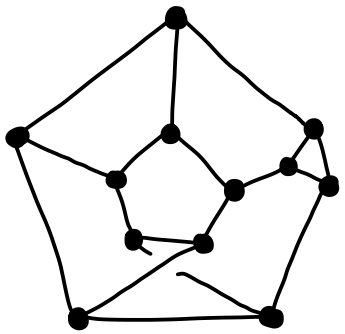
Graph Isomorphism problem

Q: Decide whether finite graphs G, H are isomorphic

Ex.

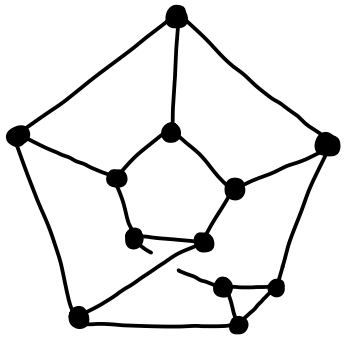
#spanning trees

$\kappa(G) = 8100$



$\not\simeq$

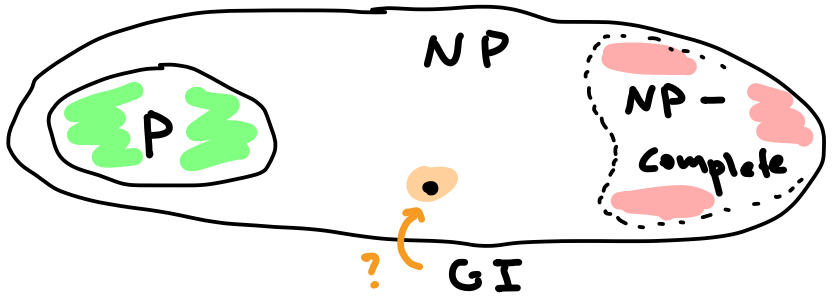
$\kappa(H) = 8131$



Graph Isomorphism: Complexity

- Exact complexity of Graph Isomorphism (GI) not known

- Conjecturally, GI is "NP-intermediate" i.e. neither P nor NP-complete



Rmk. Subgraph Isomorphism is known to be NP-complete.
↳ "Given (G, H), does G contain H as a subgraph?"

- (Babai 2016) Fastest known algo. for GI is $n^{O((\log n)^b)}$-time, for constant b

Graph Isomorphism: Complexity

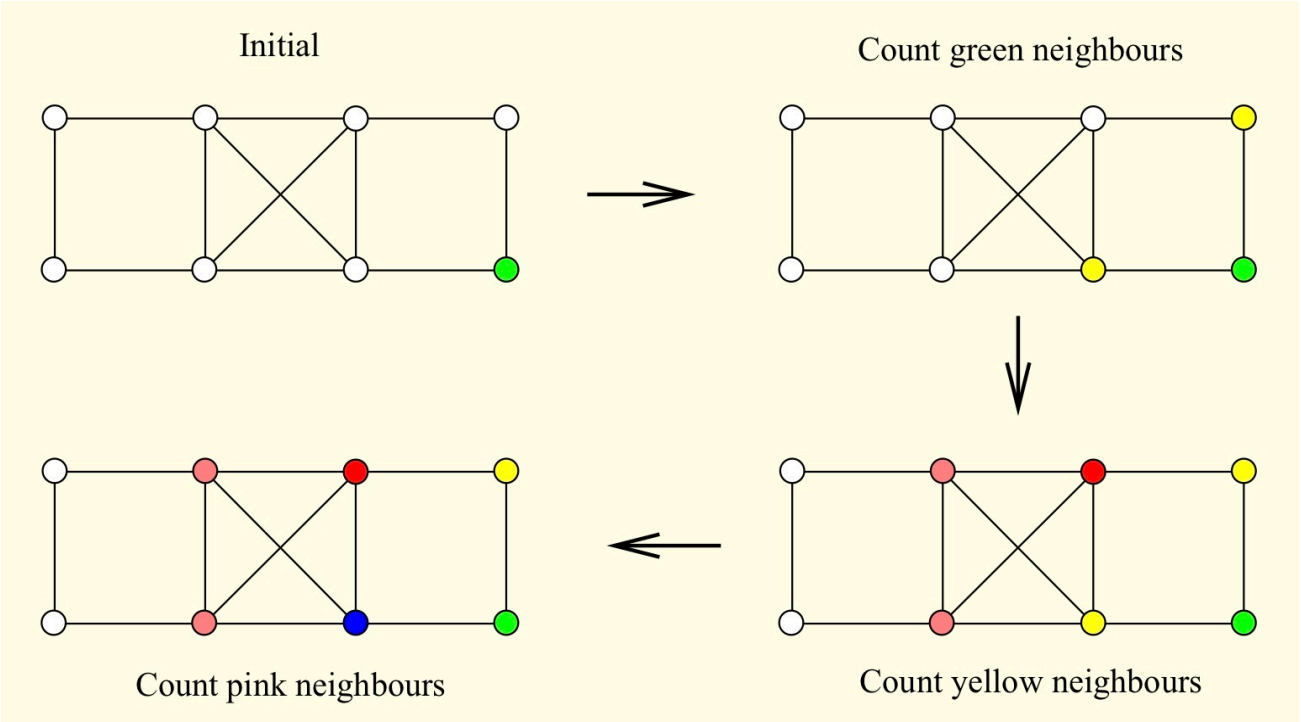
- GI known to be polynomial time on many special classes of graphs, e.g.
  - bounded degree (Grohe, Neuen, & Schweitzer 2018)
  - bounded embedding genus
    - e.g. · planar
      · torus-embeddable
  - excluded minors
  - excluded topological subgraphs

Rmk (McKay) "Few of these [polyn.-time] algorithms are practical, planar graphs being a notable exception"

Graph Isomorphism: Heuristics

- In practice, all current fastest implementations for GI in software use "canonical labelling" approach
  e.g. nauty (McKay), Traces (Piperno), bliss (Junttila & Kaski), ...

- Specifically, use "individualization-refinement" paradigm on vertices



[McKay]

Graph Isomorphism: Heuristics

The Graph Isomorphism Disease*

Ronald C. Read
UNIVERSITY OF WATERLOO

Derek G. Corneil
UNIVERSITY OF TORONTO

ABSTRACT

The graph isomorphism problem—to devise a good algorithm for determining if two graphs are isomorphic—is of considerable practical importance, and is also of theoretical interest due to its relationship to the concept of NP-completeness. No efficient (i.e., polynomial-bound) algorithm for graph isomorphism is known, and it has been conjectured that no such algorithm can exist. Many papers on the subject have appeared, but progress has been slight; in fact, the intractable nature of the problem and the way that many graph theorists have been led to devote much time to it, recall those aspects of the four-color conjecture which prompted Harary to rechristen it the "four-color disease." This paper surveys the present state of the art of isomorphism testing, discusses its relationship to NP-completeness, and indicates some of the difficulties inherent in this particularly elusive and challenging problem. A comprehensive bibliography of papers relating to the graph isomorphism problem is given.

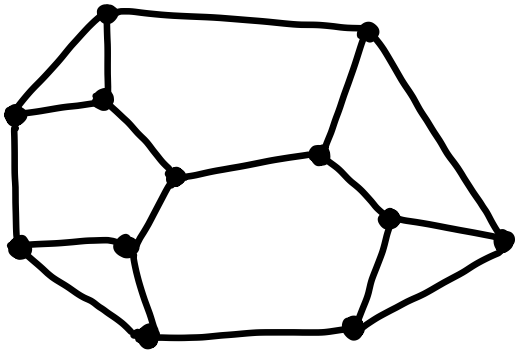
Rmk GI used to be very popular subject in theoretical computer science (1977)

Weierstrass weights

- For finite graph $G = (V, E)$, Weierstrass weights give vertex labels

$$\mu: V \to \mathbb{Z}_{\geq 0}$$

Ex.



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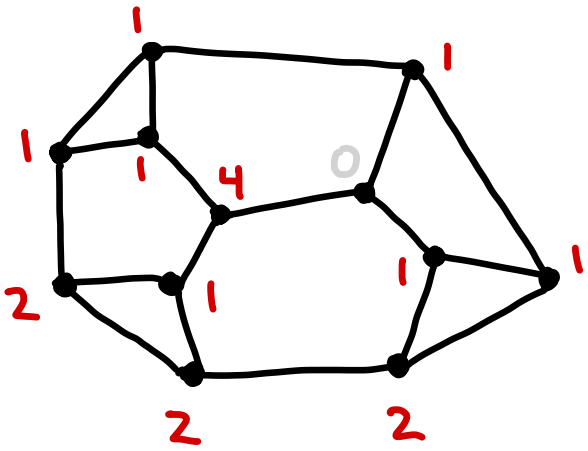

(genus = 7)

- Labels are bounded by genus := # independent cycles (a.k.a. cyclomatic #)

= # edges removed to get spanning tree = $|E| - |V| + 1$

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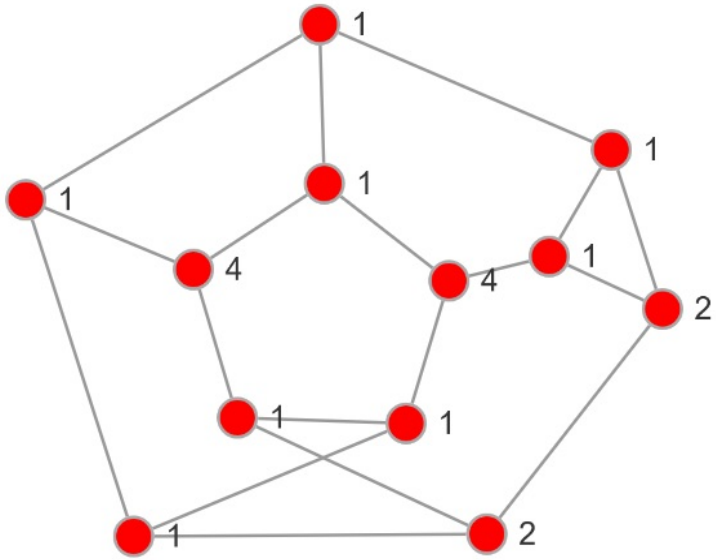

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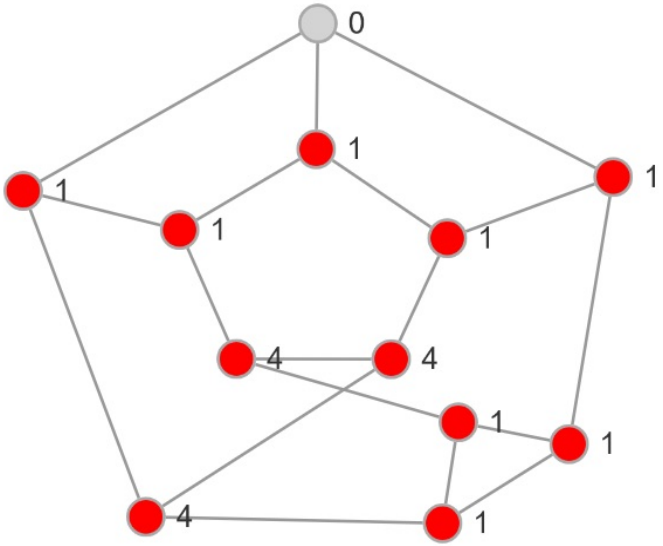
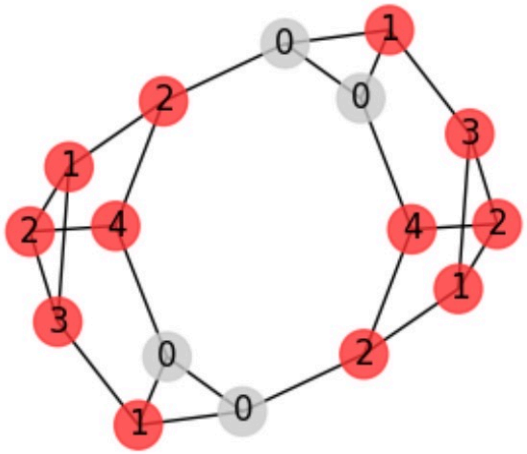
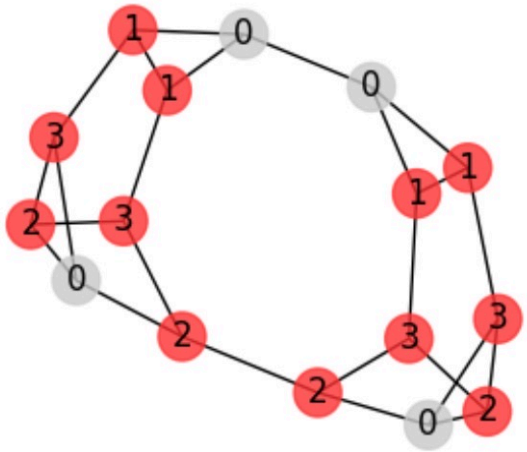


Weierstrass weights

- weights $\mu: V(G) \to \mathbb{Z}_{20}$ are isomorphism-invariant
- Claim: (informal) weights reflect mix of local & global structure

Ex. "Whitney twist" of graph two-sum





Weierstrass weights : Tropical terminology

$G = (V, E)$ finite graph

- formal integer combination of $v_i \in V$ is a **divisor**

  or "**chip configuration**"

$$D = \sum_{v_i \in V} a_i \cdot v_i \qquad \text{where} \qquad a_i \in \mathbb{Z}$$

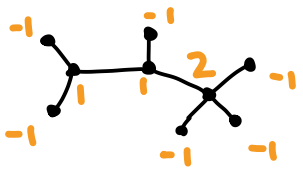
- **degree** of divisor $D = \sum_{v_i \in V} a_i \cdot v_i$ is $\deg D = \sum_{v_i \in V} a_i$

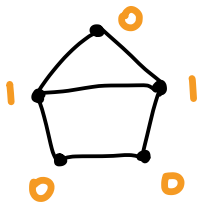
- **canonical divisor** of graph is

$$K = \sum_{v_i \in V} (\mathrm{val}(v_i) - 2)\, v_i \qquad \text{which has} \qquad \deg K = 2g - 2$$

(g: genus)

Ex.





Weierstrass weights : Dhar's burning algorithm

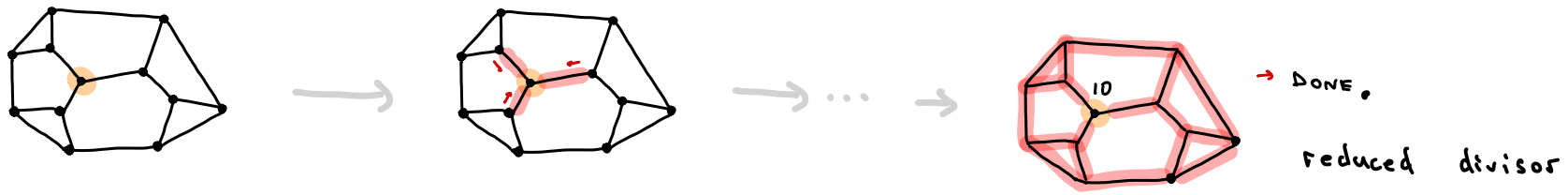
How to compute Weierstrass weights?

- For chosen vertex v.
  - Compute reduced divisor of v via Dhar's algorithm
  - weight

$$\mu(v) = (v\text{-coeff.}) - (g-1)$$

g ↳ genus

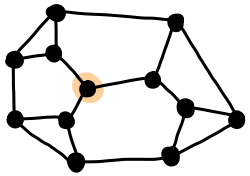
Ex.

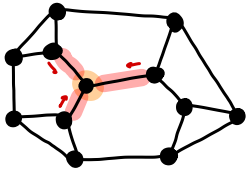


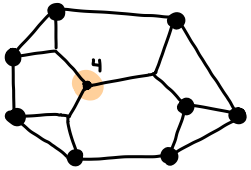
genus = 7, so $\mu(v) = 10 - 6 = 4$

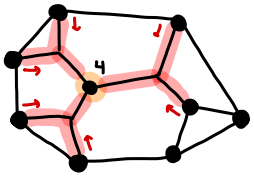
Weierstrass weights : Dhar's burning algorithm

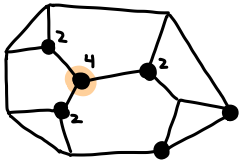
Ex.

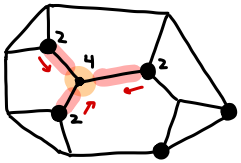






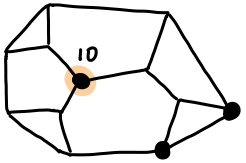


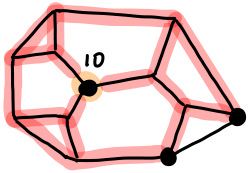


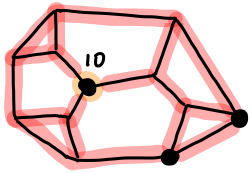


$\mu(w) = 10 - (g-1)$

$= 4$







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Weierstrass weights : Dhar's burning algorithm

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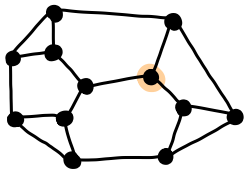
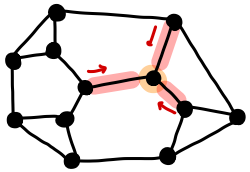
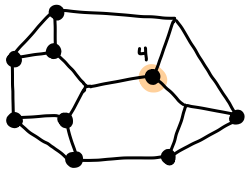
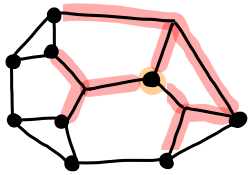
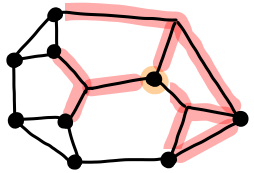
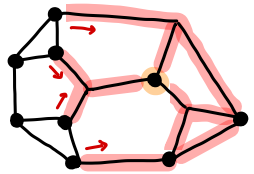


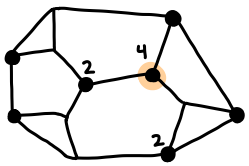


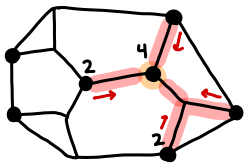






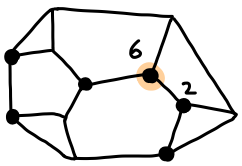


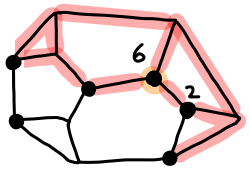


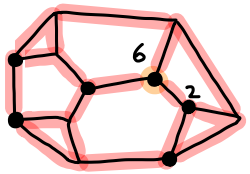


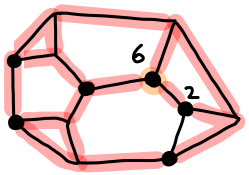
$\mu(v) = 6 - (g-1)$

$= 0$









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Weierstrass weights : Running time

- To compute Weierstrass weight at one vertex :
  1. Run "fire spreading" by BFS $O(|V| + |E|)$
  2. If fire contained on proper subgraph :
  3. Apply "chip-firing" on blocking vertices
  4. Return to 1.

  (steps 2–4) $\times O(\text{diameter})$
  5. Else, fire spreads to whole G
  6. Output chip configuration

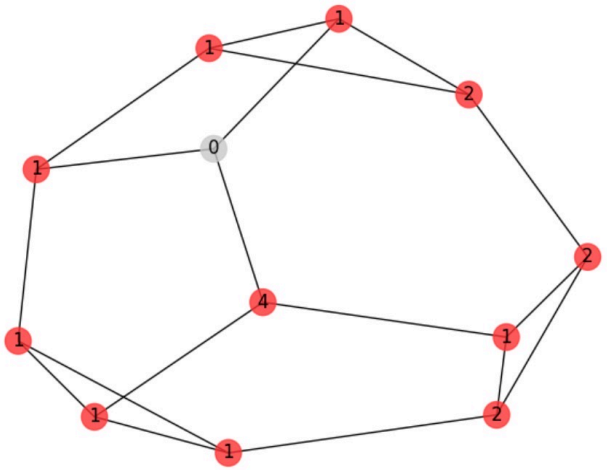
- Worst-case time $O(|V|^2 + |V| \cdot |E|)$
- Average-case for random graphs $O(|V| \log |V| + |E| \log |V|)$
- To compute weights at all vertices : don't start over

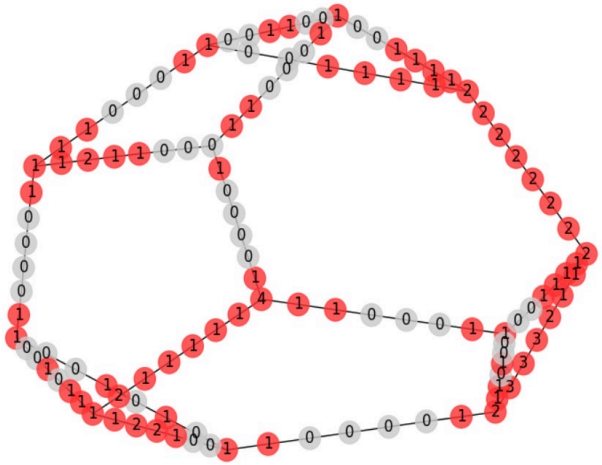
Weierstrass weights : Bounds & constraints

Prop For any vertex, Weierstrass weight $\mu(v) \leq g-1$

↳ genus

Fact. Weierstrass weights are unchanged by "uniform refinement" of edges





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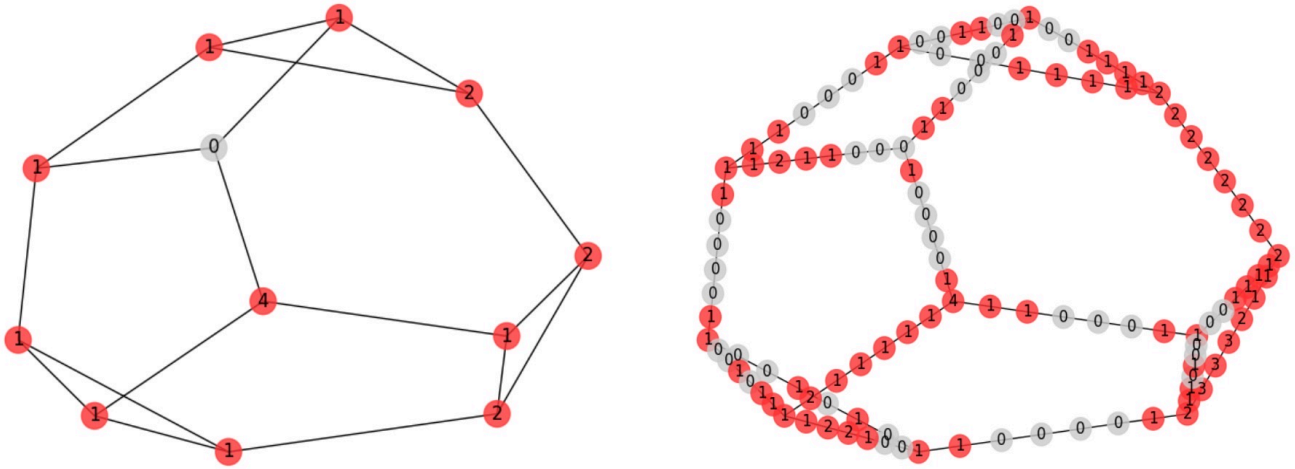

Fact Weierstrass weights are well-defined on associated metric graph, i.e. continuous space w/ edge $\simeq$ unit interval

Weierstrass weights : Bounds & constraints

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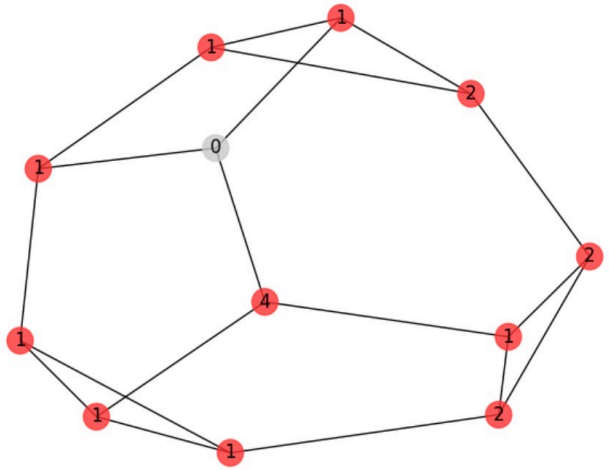
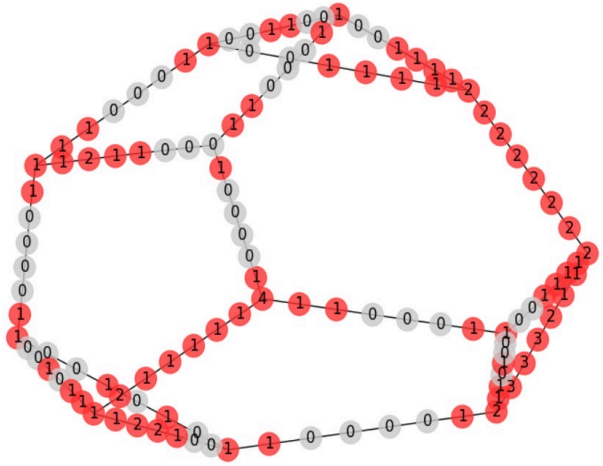
↳ genus





on underlying metric graph.

Thm (Amini - Gierczak - R 2026)

a) Finitely many connected comp. w/ "Weierstrass-condition" $\leq g^2-1$

b) "well-defined" weight on connected components, s.t.

$$\sum_i \mu(A_i) = g^2 - 1 .$$

Weierstrass weights : Bounds & constraints

on underlying metric graph.

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a) Finitely many connected comp. w/ "Weierstrass-condition"

b) "well-defined" weight on connected components, s.t.

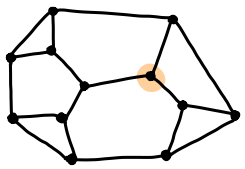
$$\sum_i \mu(A_i) = g^2 - 1 .$$

· weight formula :

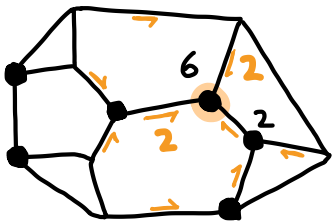
$$\mu(A) = (g+1)(g(A)-1) - \sum_{v \in \partial A} (s_0^v(K) - 1)$$

↳ "slopes" of red.-div. of K

Ex.



⇝ Reduced divisor



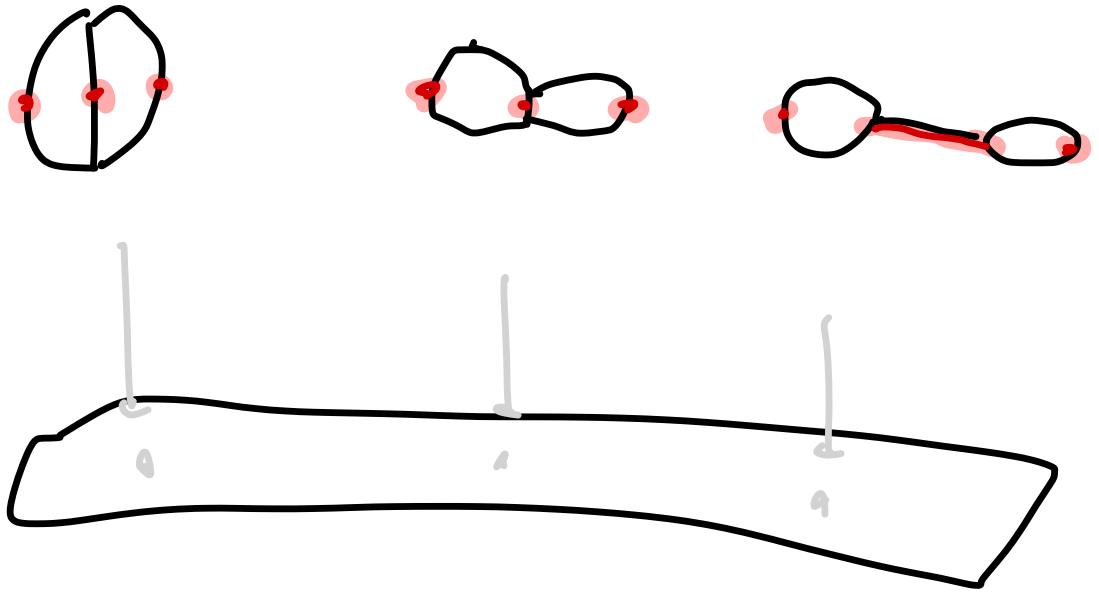
Tropical Weierstrass weights

Problem. What is "good" notion of tropical weight?

$\mathcal{G}_g^{trop}$

$\downarrow$

$\mathcal{M}_g^{trop}$



⚠ **Fact.** Tropical Weierstrass locus can be infinite, containing positive-length segments

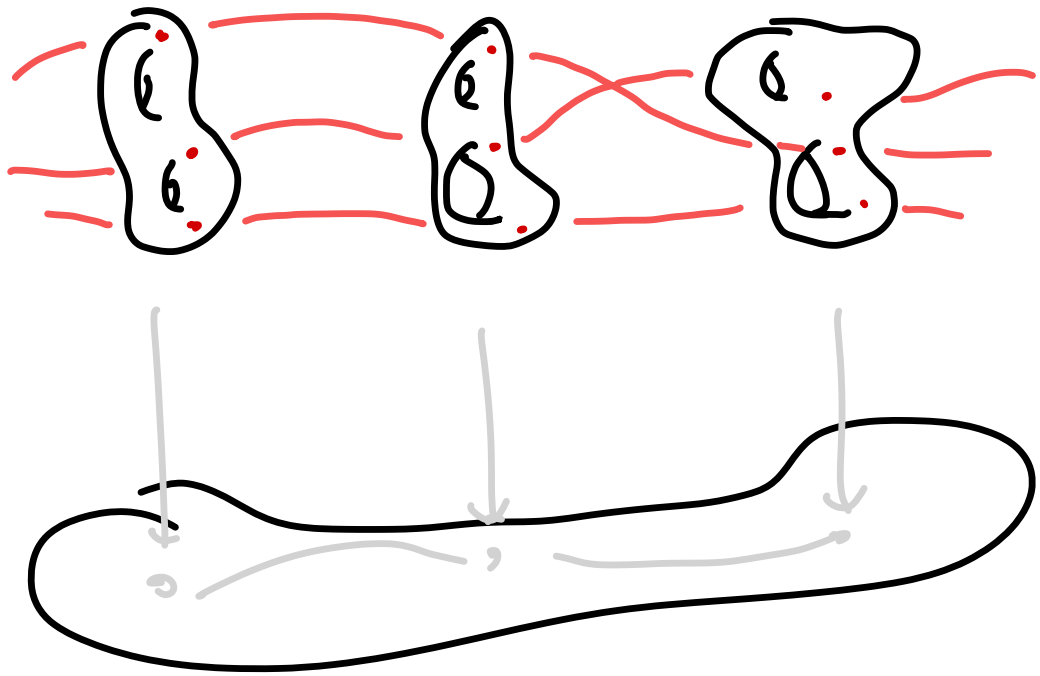
Why Weierstrass points?

Goal: understand moduli space of algebraic curves $\mathcal{M}_g$

C_g

$\downarrow$

$\mathcal{M}_g$



Slogan: "Weierstrass pts are coordinates on $\mathcal{M}_g$"

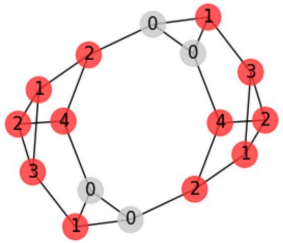
Graph isomorphism

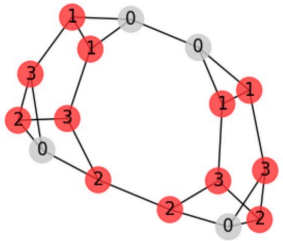
Recap:

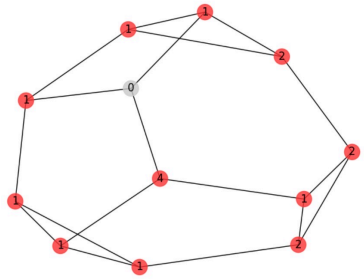
- Can Weierstrass weights improve GI running time?
  - Probably no in worst-case, arbitrary graphs
  - Maybe yes for certain families, or as heuristic
  - For highly symmetric / vertex transitive graphs, consider W-weights of edge-deletions

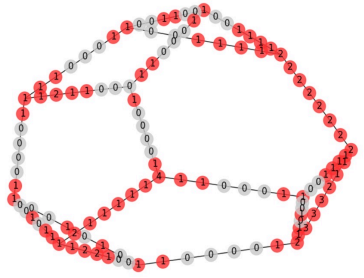
- Can we use "Jacobian" group structure on graph vertices, in other ways?

Distinguishing graphs with tropical Weierstrass weights









Thanks for listening!

Outline

- Graph isomorphism
  - problem statement
  - main results
    - quasi-polynomial time (?) Babai
- Weierstrass locus / Weierstrass weights
  - What is it ? labelling $V(G) \to \mathbb{Z}_{\geq 0}$
  - Examples ? - whitney twist
    - Vertex transitive graphs
  - Dhar's burning algorithm
- Algebraic motivation
  - algebraic curves,