Derangements & a p-adic incomplete gamma function

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arXiv: 2012.04615

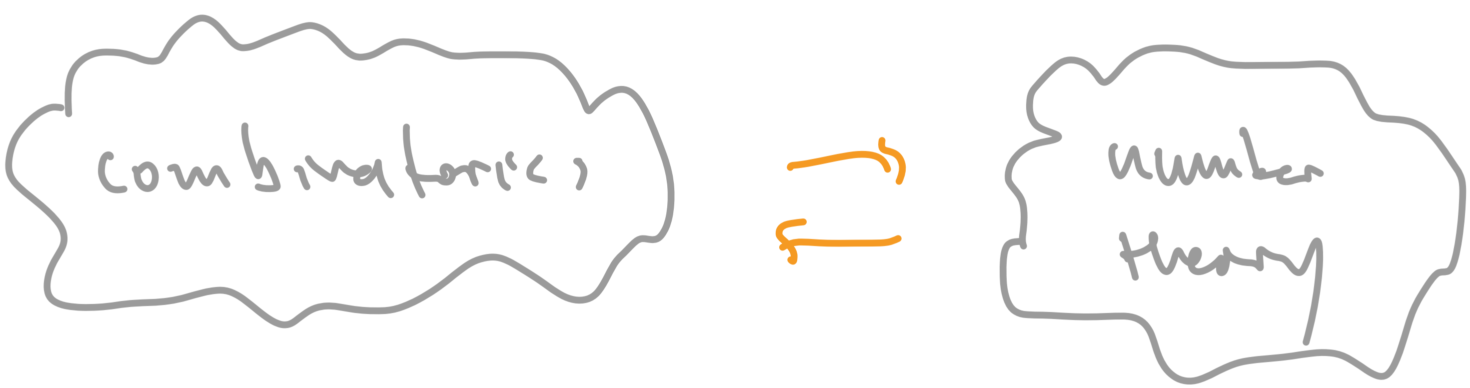
23 Nov. 2021

AAG Seminar

Why care?

- Many sequences in combinatorics obey p-adic patterns

- How can we exploit these patterns?



Derangements

A **derangement** of a finite set is a permutation w/ no fixed points

Q: How many derangements on n-element set?

n	0	1	2	3	4	5
$d(n)$	1	0	1	2	9	44

Observation: $d(n) \approx \frac{1}{e} n!$ as $n \to \infty$

(Euler 1779)

$\mathbb{R}$ - topology

Derangements

Q: How many derangements on n-element set?

n	0	1	2	3	4	5
$d(n)$	1	0	1	2	9	44

Q: How many derangements on (−1)-element set?

Observation: $n \mapsto (-1)^n d(n)$ is p-adic continuous for all p

(Sun–Zagier 2011

Miska 2016)

$\Rightarrow$ extends uniquely to $\mathbb{Z}_p \ni -1$

Derangements

p-adic continuous $\forall$ p $\Leftarrow$ ("congruence preserving" $\forall m, a, b$: $a \equiv b \pmod{m} \Rightarrow f(a) \equiv f(b) \pmod{m}$)

Q: How many derangements on n-element set?

n	0	1	2	3	4	5
$d(n)$	1	0	1	2	9	44

Q: How many derangements on (−1)-element set?

Observation: $n \mapsto \underline{(-1)^n}\, d(n)$ is p-adic continuous for $\underline{\text{all}}$ p

(Sun–Zagier 2011

Miska 2016)

$(-1)^n$ is p-adic continuous $\Leftrightarrow p = 2$

$\Rightarrow$ extends uniquely to $\mathbb{Z}_p \ni -1$

Aside: Factorials

$n! = \#(\text{permutations on } n\text{-element set})$

n	0	1	2	3	4	5
$n!$	1	1	2	6	24	120

Q: How many permutations on (-1)-element set?

Observation: $n! = \Gamma(n+1) := \int_0^\infty t^n e^{-t}\, dt$ $\mathbb{R}$-topology

(Euler 1730)

Gauss,

Weierstrass, ...

$\Rightarrow (-1)! = \Gamma(0) = \infty$

Aside: Factorials

$$n! = \#(\text{permutations on } n\text{-element set})$$

n	0	1	2	3	4	5
$n!$	1	1	2	6	24	120

Q: How many permutations on (-1)-element set?

Observation: $n \mapsto n!$ is never p-adic continuous :(

However, $n \mapsto (-1)^n \prod_{\substack{k \le n \\ (p \nmid k)}} k$ is !

(Morita 1975)

$$=: \Gamma_p(n+1) \quad (+1)$$

"Derangement - like" sequences

- arrangements of $[n]$ = choose a subset $A \subset [n]$ & permutation of A

- r-cyclic derangements in $C_r \wr S_n$

  cyclic group

- cycle-restricted permutations

"Derangement - like" sequences

- arrangements of $[n]$ = choose a subset $A \subset [n]$

  + permutation of A

  (or, linear order on A)

$$a(n) = \#\left(\begin{array}{c}\text{arrangements} \\ \text{of } [n]\end{array}\right) = \sum_{k=0}^{n} \binom{n}{k} k!$$

$$d(n) = \#\left(\begin{array}{c}\text{derangements} \\ \text{on } [n]\end{array}\right) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k!$$

inclusion-exclusion

"Derangement-like" sequences

"wreath product"

arrangements

- r-cyclic derangements in $C_r \wr S_n$ (Assaf 2010)

cyclic group   permutations

$$C_r \wr S_n = \text{"r-signed permutations on } [n]\text{"}$$

$$\cong \left(n \times n \text{ permutation matrices w/ nonzero entries} = \zeta_r^k \right)$$

↪ natural action on $[n] \times [r]$

"derangement" → no fixed pts on $[n] \times [r]$

$$d(n,r) = \sum (-1)^{n-k} r^k \binom{n}{k} k!$$

$$a(n,r) = \sum r^k \binom{n}{k} k!$$

"Derangement - like" sequences

- cycle - restricted permutations

L = set of pos. integers

$d^L(n)$ = # (permutations on $[n]$ w/ cycle lengths in L)

derangements $d(n) \longleftrightarrow L = \{2, 3, 4, \ldots\}$

factorial $n! \longleftrightarrow L = \{1, 2, 3, \ldots\}$

cycle-restricted permutations

$$\prod_{\ell=1}^{\infty} \exp\left(\frac{x^\ell}{\ell}\right) = \frac{1}{1-x}$$

L = set of pos. integers

$d^L(n) = \#$ (permutations on $[n]$ w/ cycle lengths in L)

Q: How to compute $d^L(n)$?

(generating f'ns) + (combinatorial species)

$$\Rightarrow \left[\sum_{k\geq 0} d^L(k)\frac{t^k}{k!} = \prod_{\ell \in L} \exp\left(\frac{t^\ell}{\ell}\right) \right]$$

derangements $\displaystyle\sum d(k)\frac{t^k}{k!} = \prod_{\ell\geq 2}\left(\text{''} \right) = \frac{e^{-t}}{1-t}$

cycle-restricted permutations

L = set of pos. integers

$d^L(n) = \#\left(\text{permutations on } [n] \text{ w/ cycle lengths in } L\right)$

Q: How to compute $d^L(n)$?

(generating fns) + (combinatorial species)

$$\Rightarrow \sum_{k\geq 0} \underbrace{d^L(k)}_{\text{p-adic continuous?}} \frac{t^k}{k!} = \prod_{\ell\in L} \exp\left(\frac{x^\ell}{\ell}\right)$$

How to detect p-adic continuity?

Def The Mahler series for $f(x) : \mathbb{N} \to \mathbb{Q}_p$ is

$$f(x) = \sum_{k \geq 0} c_k \binom{x}{k} = c_0 + c_1 \binom{x}{1} + c_2 \binom{x}{2} + \cdots$$

where

$$\binom{x}{k} := \frac{1}{k!} \underbrace{x(x-1)(x-2)\cdots(x-k+1)}$$

$\frac{1}{k!} x^{\underline{k}}$ "falling factorial"

Ex. $3x^2 + 5x + 1 = 1 + 8\binom{x}{1} + 6\binom{x}{2}$

$$3^x = (1+2)^x = 1 + 2\binom{x}{1} + 2^2\binom{x}{2} + 2^3\binom{x}{3} + \cdots$$

Binomial Thm

How to detect p-adic continuity?

Def The Mahler series for $f(x): \mathbb{N} \to \mathbb{Q}_p$ is

$$f(x) = \sum_{k \geq 0} c_k \binom{x}{k} = c_0 + c_1 \binom{x}{1} + c_2 \binom{x}{2} + \cdots$$

Thm (Mahler 1958)

$f(x)$ p-adic continuous $\iff$ $|c_k|_p \to 0$ as $k \to \infty$

Lipschitz continuous w/ constant C $\iff$ $\sup_{k \geq 1} |c_k|_p \, k \leq C$

Amice: 1964 locally analytic $\iff$ $\limsup_{k \to \infty} |c_k|_p^{1/k} < 1$

How to detect p-adic continuity?

$$f(x) = \sum_{k\geq 0} c_k \binom{x}{k} = c_0 + c_1\binom{x}{1} + c_2\binom{x}{2} + \cdots$$

Thm (Mahler 1958)

$f(x)$ p-adic continuous $\iff$ $|c_k|_p \to 0$ as $k \to \infty$

Q: How to find Mahler coefficients c_k?

$$c_k = \Delta^k f(0) = f(k) - \binom{k}{1} f(k-1) + \binom{k}{2} f(k-2) + \cdots + (-1)^k f(0)$$

$$= \sum_{i=0}^{k} (-1)^i \binom{k}{i} f(k-i)$$

k-th finite difference

Finite difference

$\Delta f(x) = f(x+1) - f(x)$

How to detect p-adic continuity?

Q: How to find Mahler coefficients c_k?

$$c_k = \Delta^k f(0) = f(k) - \binom{k}{1} f(k-1) + \binom{k}{2} f(k-2) + \cdots + (-1)^k f(0)$$

k-th finite difference

$$= \sum_{i=0}^{k} (-1)^i \binom{k}{i} f(k-i)$$

$\Rightarrow$ apply exponential generating functions

$$c_k = \sum_{i=0}^{k} (-1)^i \binom{k}{i} f(k) \iff e^{-x} \sum_{k=0}^{\infty} f(k) \frac{t^k}{k!} = \sum_{k=0}^{\infty} c_k \frac{t^k}{k!}$$

$$\left(\text{equiv.} \quad \sum \binom{k}{i} c_k = f(k) \right)$$

$$\sum f(k) \frac{t^k}{k!} = e^x \sum c_k \frac{t^k}{k!}$$

p-adic continuity of cycle-restricted permutations

Theorem (O'Desky - R)

$$d^L(n) = \#\left(\begin{array}{l}\text{permutations on } [n] \text{ w/} \\ \text{cycle lengths in } L\end{array}\right)$$

1.) If $1 \in L$, then $n \mapsto d^L(n)$ is p-adic continuous iff $p \notin L$

(and $(-1)^n d^L(n)$ not p-adic continuous for $p \geq 3$)

2.) If $1 \notin L$, then $n \mapsto (-1)^n d^L(n)$ is p-adic continuous iff $p \in L$.

($d^L(n)$ not p-adic continuous for $p \geq 3$)

p-adic continuity of cycle-restricted permutations

Theorem (O'Desky - R)

1.) If $1 \in L$, then $n \mapsto d^L(n)$ is p-adic continuous iff $p \notin L$

2.) If $1 \notin L$, then $n \mapsto (-1)^n d^L(n)$ is p-adic continuous iff $p \in L$.

Pf sketch Check that EGF coeffs of $\exp\left(\frac{x^m}{m}\right)$ decay p-adically $\iff$ $m \neq 1$ or p.

Incomplete gamma function

Gamma function $\Gamma(s) = \int_0^\infty t^s e^{-t} \frac{ds}{s}$

Incomplete gamma function (Upper) $\Gamma(s,z) = \int_z^\infty t^s e^{-t} \frac{ds}{s}$

* "extra" factor

Observation: $\Gamma(n+1, -1/r) = \#(\text{derangements in } S_n \wr C_r) \cdot \overbrace{e^{-1/r}}$

Theorem (O'Desky - R) $\exists$ p-adic continuous function

$\Gamma_p : \mathbb{Z}_p \times (1 + p\mathbb{Z}_p) \to \mathbb{Z}_p$ which interpolates *

$(n, r) \mapsto \Gamma(n+1, r)$

Derangements of size -1:

$$d(-1) = -\sum_{k\geq 0} k! = -(0! + 1! + 2! + 3! + \cdots) \in \mathbb{Z}_p$$

in $\mathbb{R}$?

$$\overset{?}{=} (-1)^{?}\, 0.697\ldots$$

Euler: $\displaystyle\sum_{n=0}^{\infty} (-1)^n n! = 0.596$?

(see Lagarias "Euler's constant: Euler's work and modern developments")