Looking for a “Local” Gauss-Lucas Theorem

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Locating roots

Problem

How do roots of $f(x)$ determine roots of $f'(x)$?

Locating roots: real case

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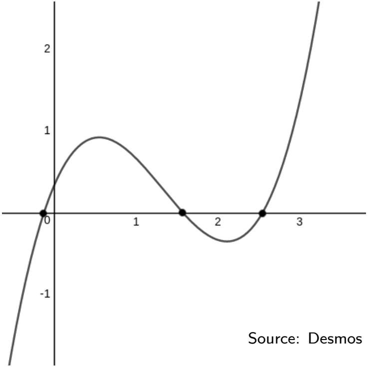

Source: Desmos

$f(x) \in \mathbb{R}[x]$

Locating roots: real case

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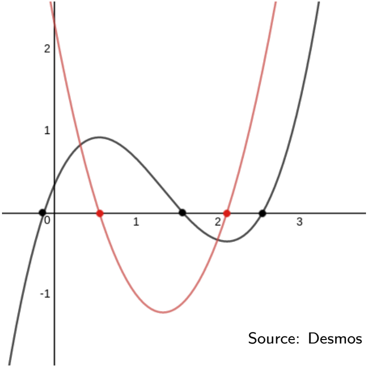

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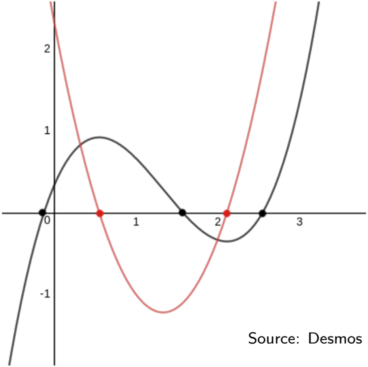


$f(x) \in \mathbb{R}[x]$

Theorem (Rolle)

Suppose $f(x) \in \mathbb{R}[x]$ with real roots $a_1 \leq \cdots \leq a_n$. Then for any $i < j$, the closed interval

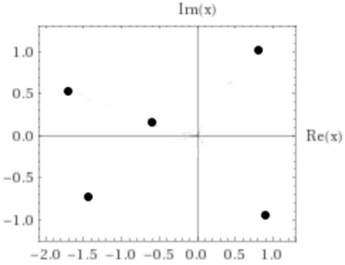
$$I = [a_i, a_j] \subset \mathbb{R}$$

contains some root of $f'(x)$.

Locating roots: complex case

Problem

How do roots of $f(z)$ determine roots of $f'(z)$?



$$f(z) \in \mathbb{C}[z]$$

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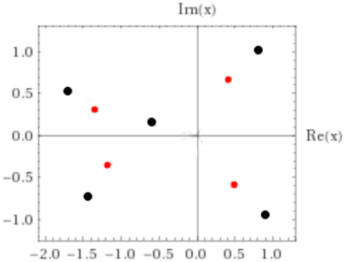

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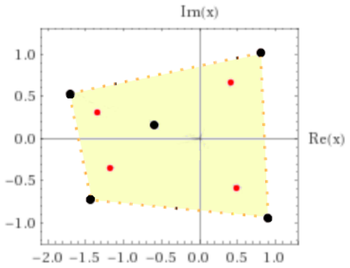

$f(z) \in \mathbb{C}[z]$

Theorem (Gauss-Lucas)

Suppose $f(z) \in \mathbb{C}[z]$ with roots $a_1, \ldots, a_n$, and convex hull

$$K = K(a_1, \ldots, a_n) \subset \mathbb{C}.$$

Then all roots of $f'(z)$ lie inside K.

Locating roots: comparison

Theorem (Rolle)

Suppose $f(x) \in \mathbb{R}[x]$ *with real roots* $a_1 \leq \cdots \leq a_n$. *Then for any* $i < j$, *the closed interval*

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- local condition

Locating roots: comparison

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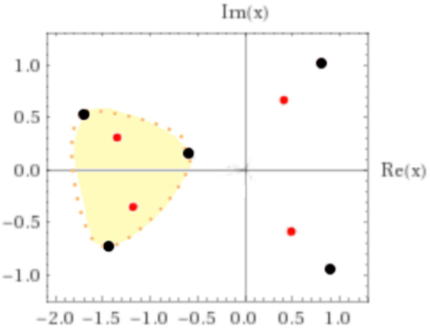
Then K contains all roots of $f'(z)$.

- local condition
- non-local condition

Local Gauss-Lucas

Conjecture

For $f(z) \in \mathbb{C}[z]$ with roots at $a_1, a_2, a_3 \in \mathbb{C}$, there is a compact region $I(a_1, a_2, a_3) \subset \mathbb{C}$ such that $f'(z)$ has a root inside $I(a_1, a_2, a_3)$.



Local Gauss-Lucas: guesses?

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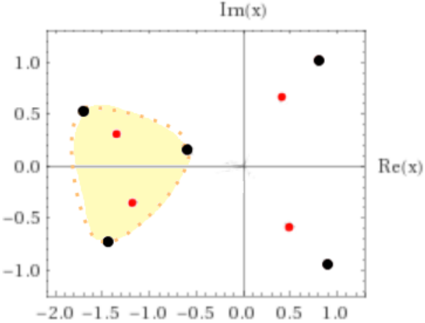

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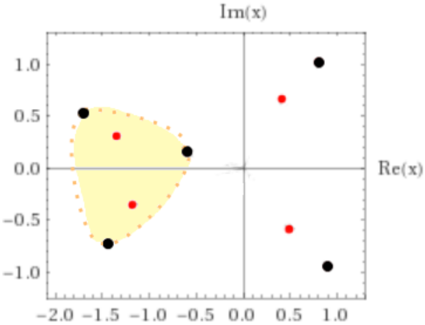
Guess 1

$I(a_1, a_2, a_3) =$ convex hull.



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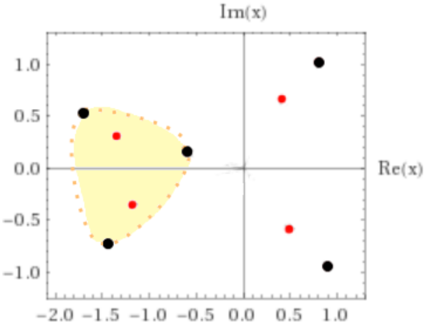
Guess 1

$I(a_1, a_2, a_3) =$ convex hull.

Guess 2

$I(a_1, a_2, a_3) =$ circumcircle.



Local Gauss-Lucas: false guess

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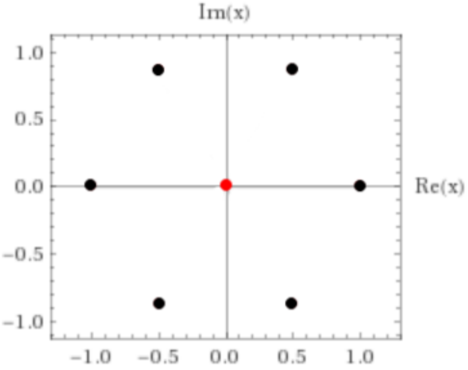

$$f(z) = z^6 - 1$$

Local Gauss-Lucas: false guess

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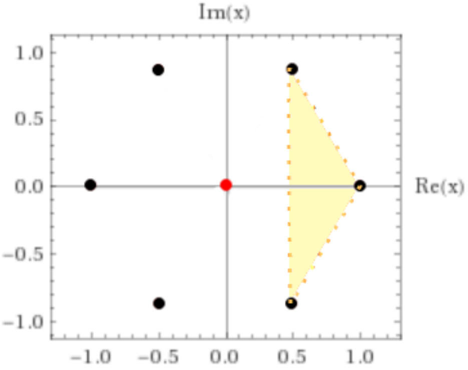

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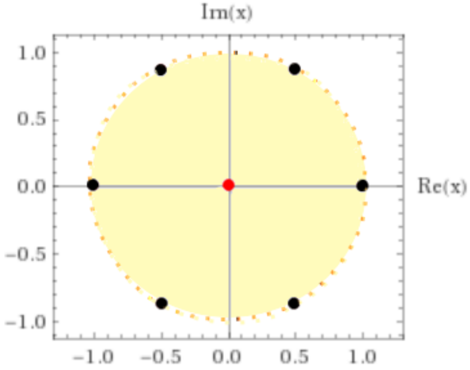
Guess 2

$I(a_1, a_2, a_3) =$ circumcircle.



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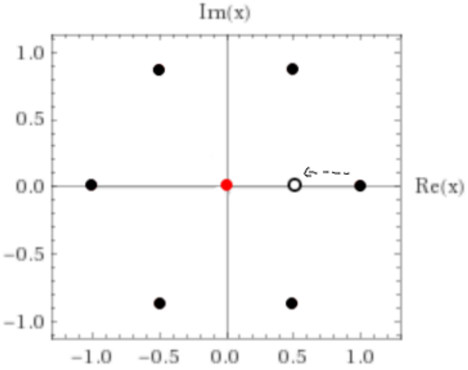
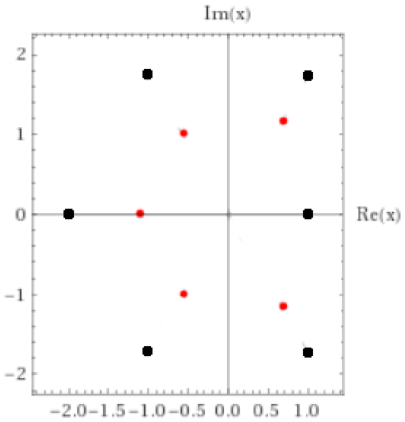


$$f(z) = (z^6 - 1) \cdot \frac{z - 1/2}{z - 1}$$

Guess 2

$I(a_1, a_2, a_3) =$ circumcircle.



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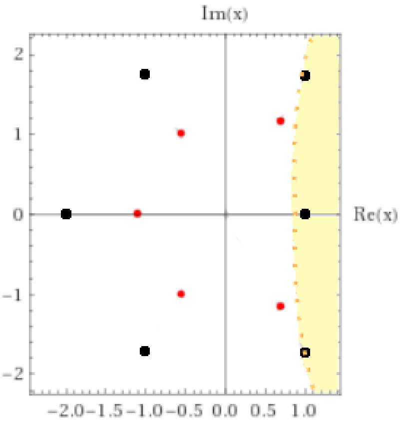
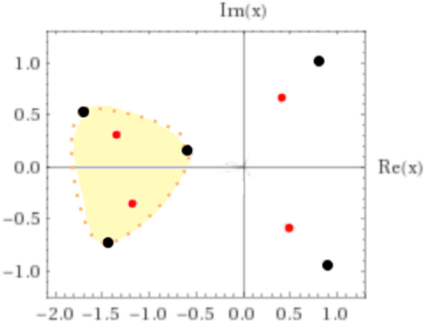


$$f(z) = (z^6 - 1) \cdot \frac{z-1/2+\epsilon}{z-1}$$

Local Gauss-Lucas: more guesses?

Conjecture

For $f(z) \in \mathbb{C}[z]$ with roots at $a_1, a_2, a_3 \in \mathbb{C}$, there is a compact region $I(a_1, a_2, a_3) \subset \mathbb{C}$ such that $f'(z)$ has a root inside $I(a_1, a_2, a_3)$.



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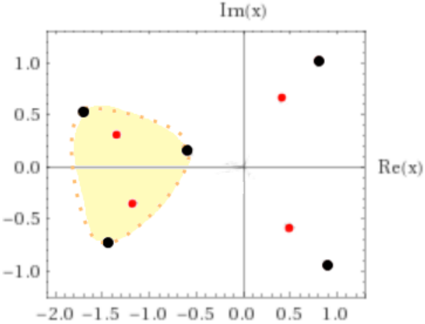
Guess 3

$I(a_1, a_2, a_3) = 2 \cdot (\text{circumcircle})$.



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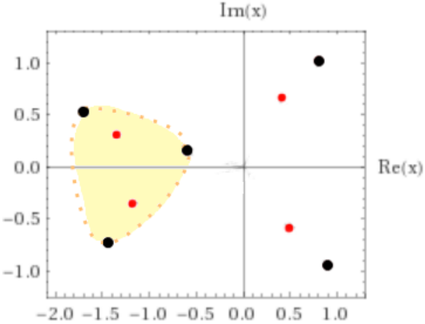
Guess 3

$I(a_1, a_2, a_3) = 2 \cdot (\text{circumcircle})$.

Guess 4

$I(a_1, a_2, a_3) =$ circumcircle, if they span **acute** triangle.



References

Harry Richman (2017)
"Local" Gauss-Lucas theorem?,
MathOverflow, https://mathoverflow.net/q/262906

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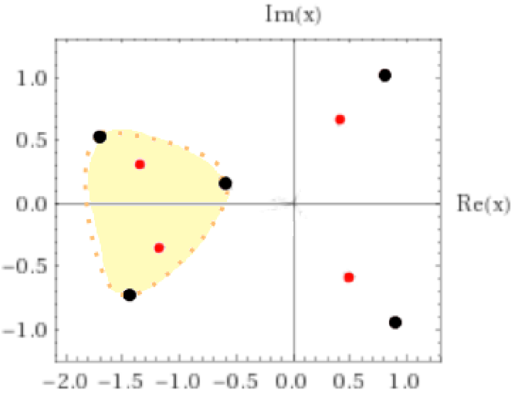

Source: WolframAlpha

Thank you!