Planar pentagons & K3 surfaces

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國立中正大學

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Harry Richman

NCTS, Taipei

joint w/ Flora Poon

K3 Automorphisms

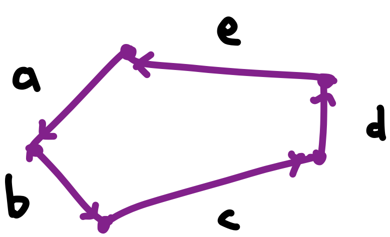
<u>Theorem</u> (Oguiso, 2003) For a local family of K3 surfaces,

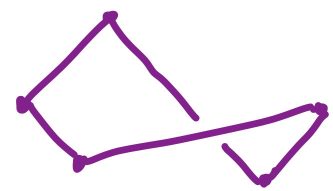
a) for fibers X_t where Picard rank is generic, $\mathrm{Aut}(X_t)$ can only "jump up"

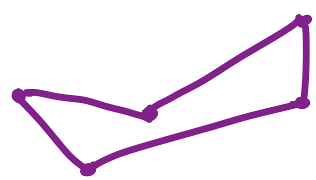
b) for special fibers X_t where Picard rank jumps, $\mathrm{Aut}(X_t)$ can shrink

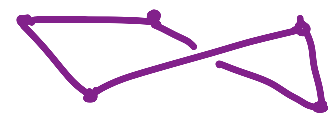
Planar pentagons

- Fix tuple of side lengths $(a_1, \dots, a_5) \in \mathbb{R}^5_{>0}$
- Let angles vary



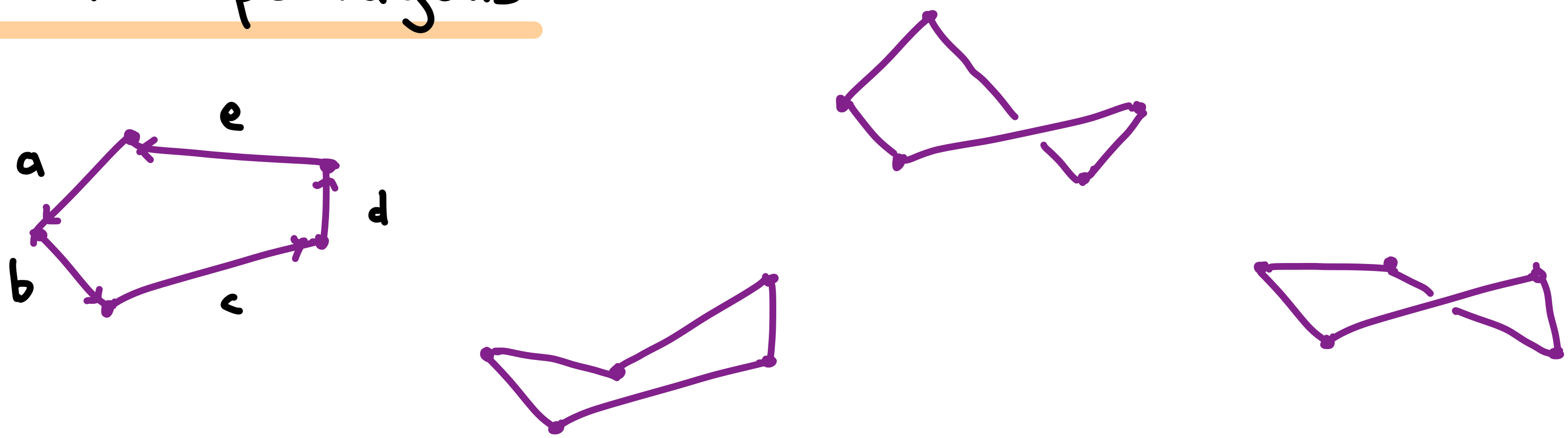






$$X_{(a_1,\dots,a_5)} = \{ \text{labeled, oriented planar pentagons up to } \mathrm{Aut}_+(\mathbb{R}^2) \}$$

Planar pentagons



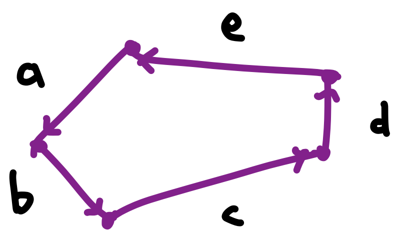
<u>Theorem</u> (Kapovich – Millson, Jaggi, Walker, ...)

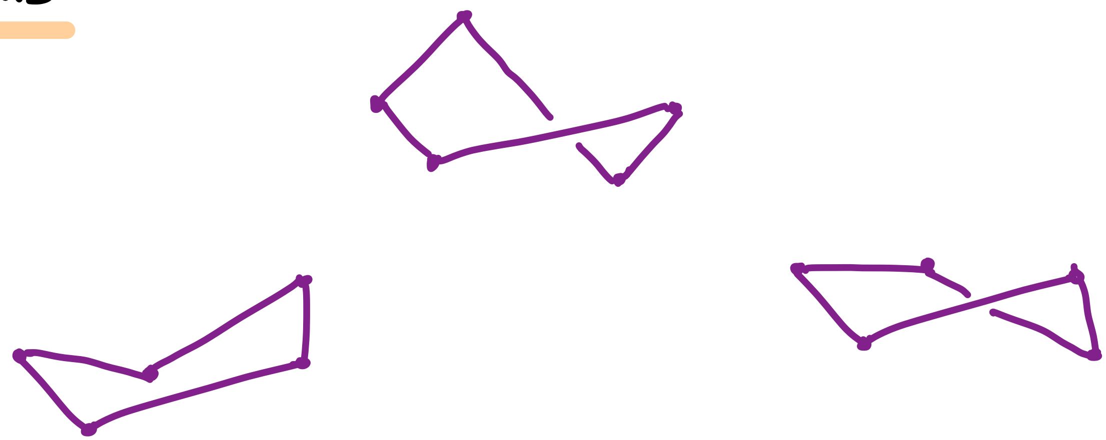
If non-empty, $X_{(a,b,c,d,e)}$ is a 2-dim smooth manifold iff

$$0 \neq a \pm b \pm c \pm d \pm e.$$

- For "equilateral" pentagons, $X_{(1,\dots,1)}$ is smooth genus 4 surface
- Also, describes singularities in non-smooth case

Planar pentagons





Theorem (Cantat - Dujardin, 2023)

$X_{(a,b,c,d,e)}$ = $\mathbb{R}$ - points of K3 surface,

smooth iff

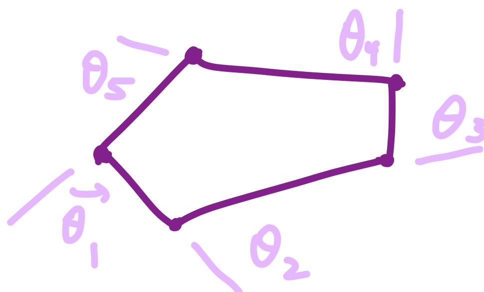
$$0 \neq a \pm b \pm c \pm d \pm e. \quad (*)$$

$\Rightarrow$ family of K3 surfaces

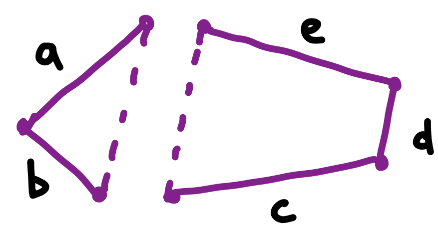
$$\{ X_{(a,\dots,e)} \mid (a,b,c,d,e) \in \mathbb{R}^5, \text{ non-degenerate}^{*} \}$$

"pentagon surfaces"

Planar pentagons: algebraic structure



cut along diagonal



- Trigonometry:

$$a^2 + b^2 + 2ab\cos\theta_1 = c^2 + d^2 + e^2 + 2cd\cos\theta_3 + 2de\cos\theta_4 + 2ce\cos(\theta_3+\theta_4)$$

- Substitute $t = \tan(\theta/2)$: $\left[\cos\theta = \frac{1-t^2}{1+t^2}, \quad \sin\theta = \frac{2t}{1+t^2}\right]$

$$(a-b)^2 + \frac{4ab}{1+t_1^2} = (c-d+e)^2 + \frac{4cd}{1+t_3^2} + \frac{4de}{1+t_4^2} + \frac{4ce(t_3+t_4)^2}{(1+t_3^2)(1+t_4^2)}$$

Planar pentagons: algebraic structure

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$$+ \frac{4de}{1+t_4^2} + \frac{4ce\,(t_3+t_4)^2}{(1+t_3^2)(1+t_4^2)} \qquad (\star_1)$$

$\Rightarrow$ Pentagon surface $X_{(a,\dots,e)} \subset (\mathbb{P}^1)^5$

cut out by $(\star_1, \dots, \star_5, \dots)$

Planar pentagons: algebraic structure

Def'n Pentagon surface $X_{(a,\dots,e)} \subset (\mathbb{P}^1)^5$

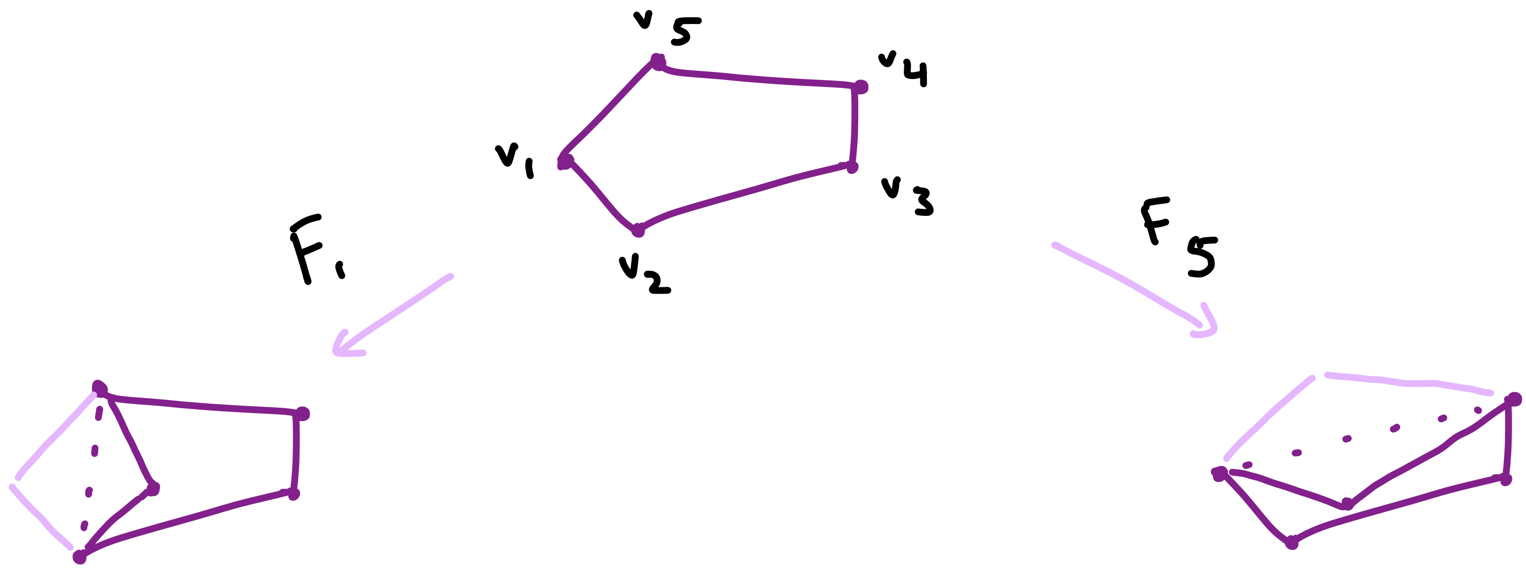
cut out by $(\star_1, \dots, \star_5, \dots)$

$$(a-b)^2 + \frac{4ab}{1+t_1^2} = (c-d+e)^2 + \frac{4cd}{1+t_3^2} + \frac{4de}{1+t_4^2} + \frac{4ce(t_3+t_4)^2}{(1+t_3^2)(1+t_4^2)} \qquad (\star_1)$$

Rmk.

- Equation $(\star_1)$ is degree $(2,2,2,2,2)$ in $t_1, \dots, t_5$
- Projection $X_{(a,\dots,e)} \subset (\mathbb{P}^1)^5 \twoheadrightarrow (\mathbb{P}^1)^3$ is birational on image
- Generic deg. $(2,2,2)$ hypersurface in $(\mathbb{P}^1)^3$ is K3

Planar pentagons: flip automorphisms

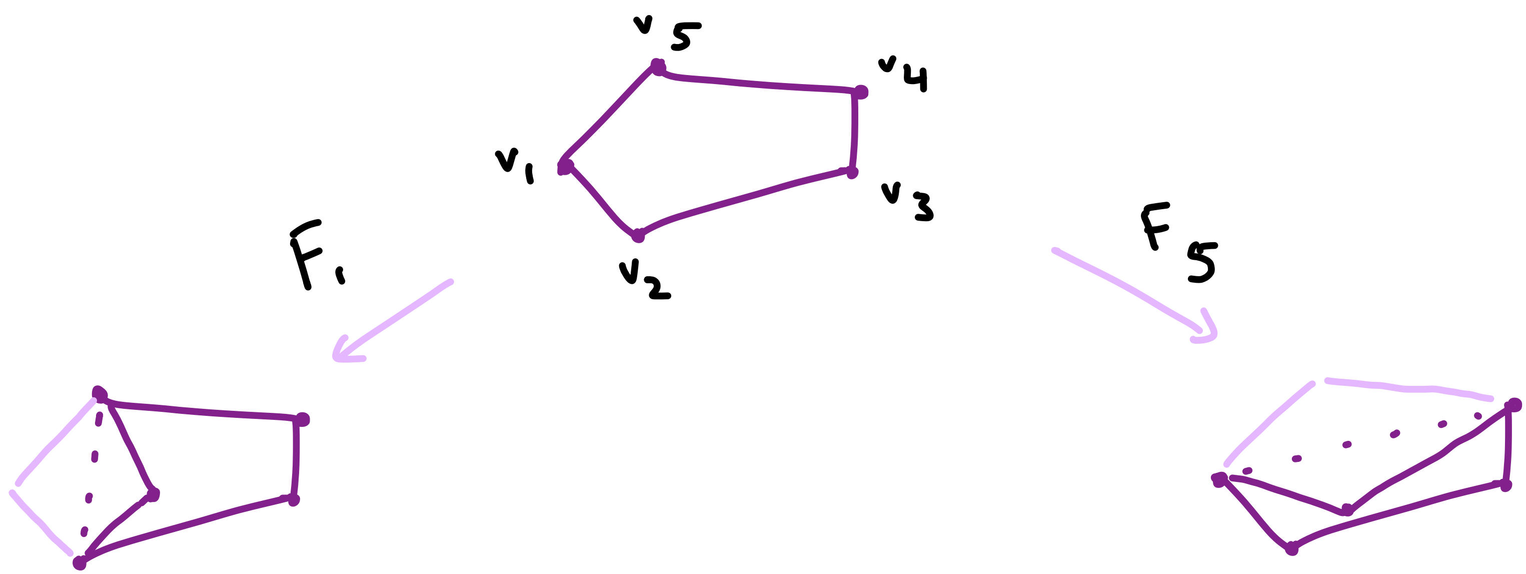


Flip operations satisfy:

- $F_i^2 = 1$
- $F_i F_j = F_j F_i$ if $|i-j| \geq 2$
- $(F_i F_{i+1})^n \neq 1$ for all $n \geq 1$,

  if side lengths $(a, \ldots, e)$ generic

Planar pentagons: flip automorphisms



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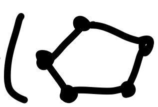
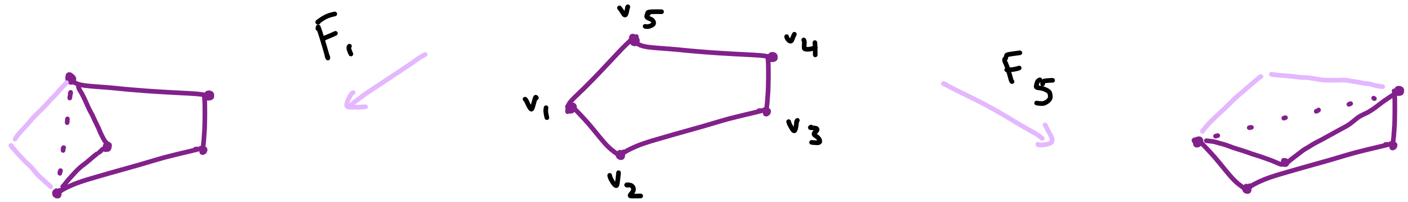
Coxeter group

Coxeter(),

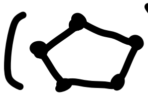
... other relations?

K3 automorphisms of pentagon surfaces



Conjecture

(i) For general side lengths, $\mathrm{Aut}(X_{(a,\dots,e)})$ contains

Coxeter() as subgroup of finite index.

infinite, hyperbolic

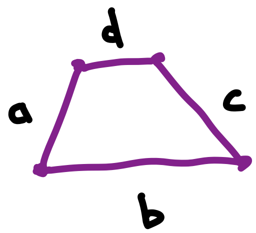
(ii) For equal side lengths, $\mathrm{Aut}(X_{(1,\dots,1)})$ contains

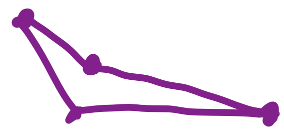
Sym_5 as subgroup of finite index.

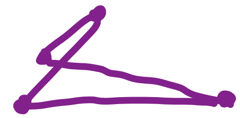
finite

Q: What are Picard ranks of $X_{(a,\dots,e)}$?

Aside: Planar quadrilaterals







Theorem (Izmestiev 2018) Planar 4-gons w/ side lengths (a, b, c, d) are parametrized by the elliptic curve*

*possibly degenerate

$$E_{\lambda} = \{ y^2 = x(x-1)(x-\lambda) \} \subset \mathbb{P}^2$$

where

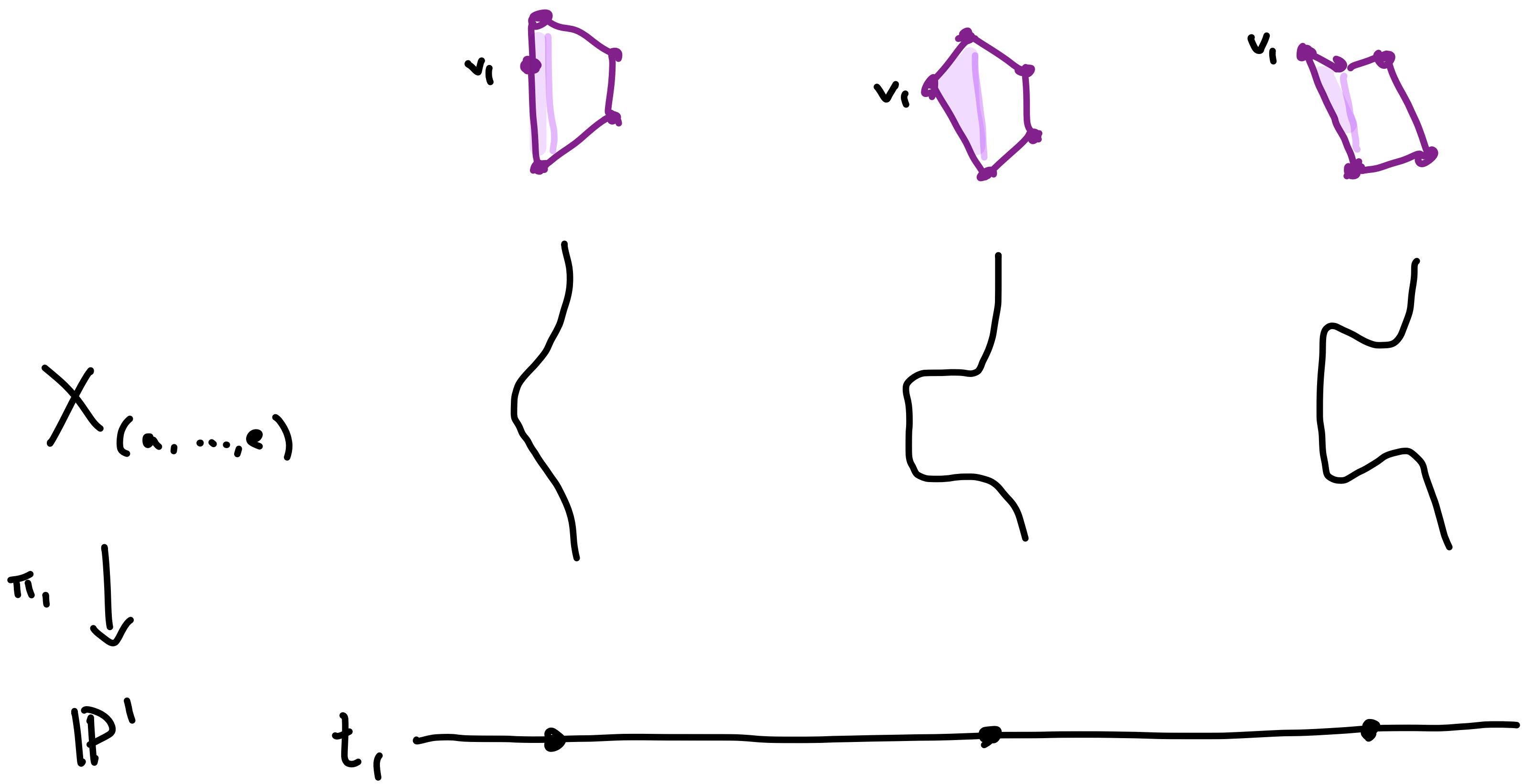
$$\lambda = \frac{abcd}{(s-a)(s-b)(s-c)(s-d)} \quad \text{and} \quad s = \tfrac{1}{2}(a+b+c+d).$$

"semi-perimeter"

Elliptic fibration

• Projection $X_{(a,\dots,e)} \subset (\mathbb{P}^1)^5 \xrightarrow{\pi_i} \mathbb{P}^1$

remembers angle i, forgets others



Elliptic fibration

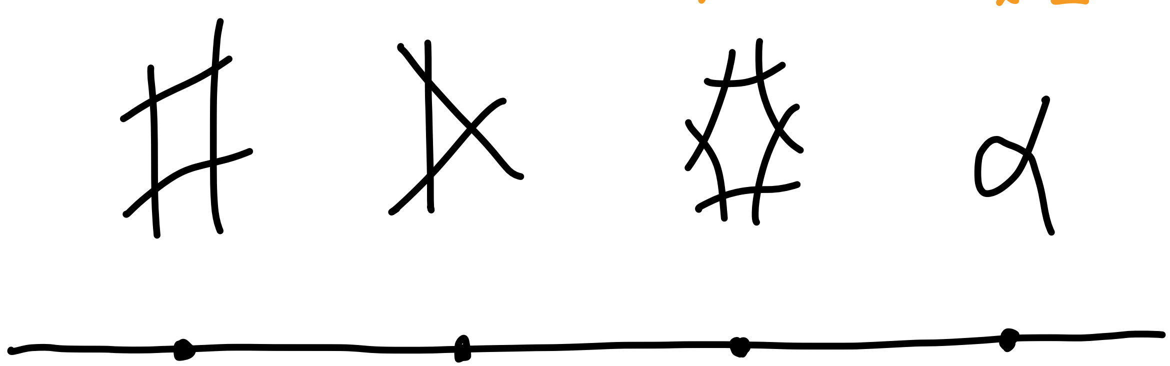
Prop. For equilateral pentagons, parametrizing surface has elliptic fibration w/ singular fibers

$$2 I_1 + 2 I_3 + I_4 + 2 I_6 \,, \quad \text{so}$$

Picard rank is $\geq 19 = 17 + 2$, Shioda-Tate

$$X_{(1,1,1,1,1)} \longrightarrow \mathbb{P}^1$$

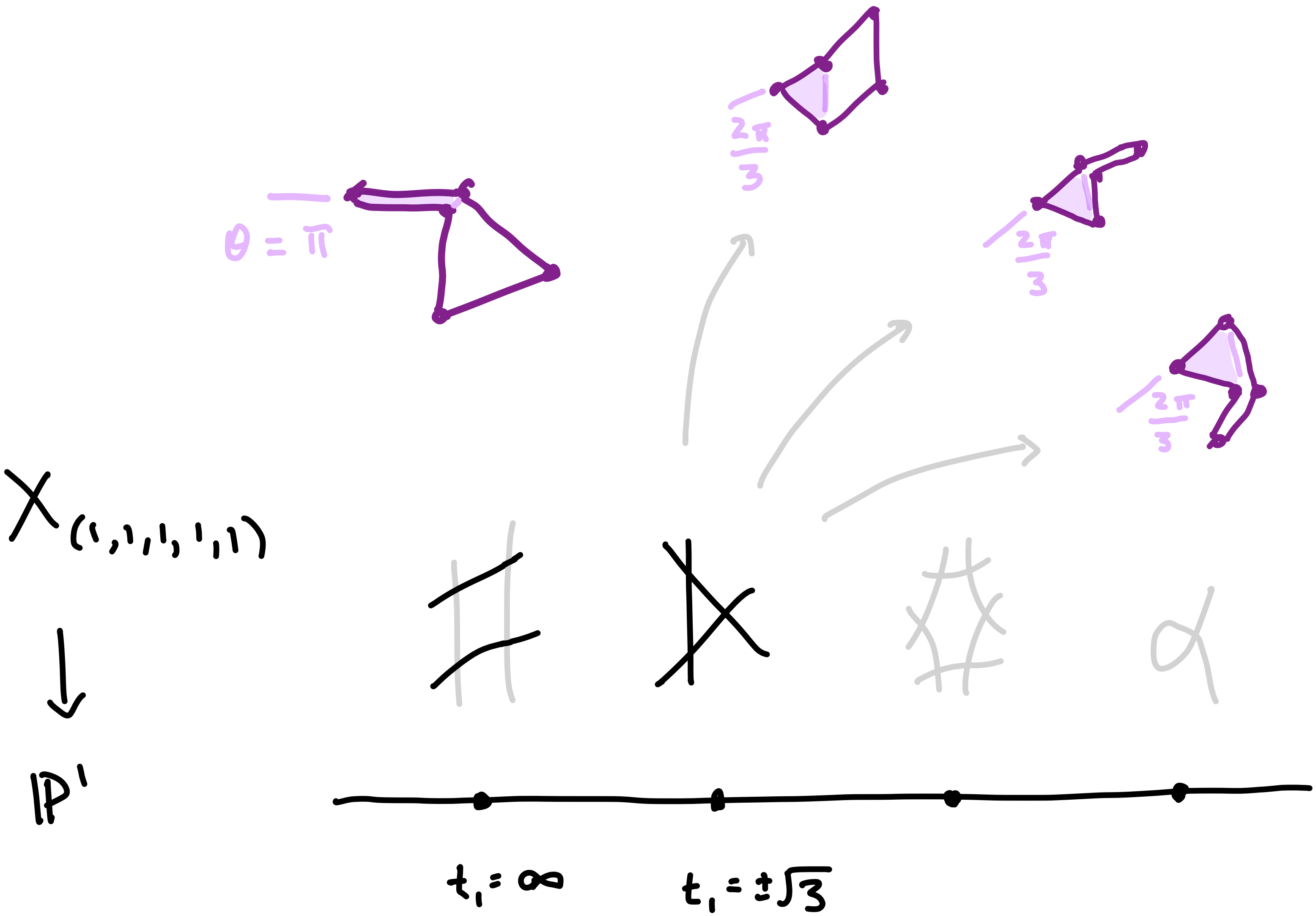
$\times 1$ $\quad \times 2$ $\quad \times 2$ $\quad \times 2$



$t_1 = \infty \qquad t_1 = \pm\sqrt{3} \qquad t_1 = \pm i \qquad t_1 = \pm \frac{\sqrt{5}}{3} i$

Elliptic fibration

Q: Which fibers are real?



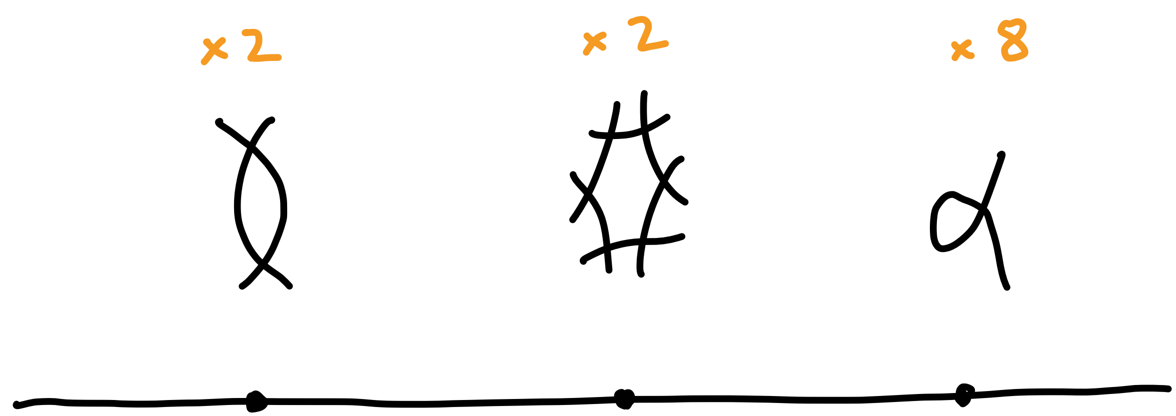
Elliptic fibration

Prop. For generic-length pentagons, parametrizing surface has elliptic fibration w/ singular fibers

$$8 I_1 + 2 I_2 + 2 I_6, \quad \text{so}$$

Picard rank is $\geq 14 = 12 + 2$

$$X_{(a,b,c,d,e)} \downarrow \mathbb{P}^1$$



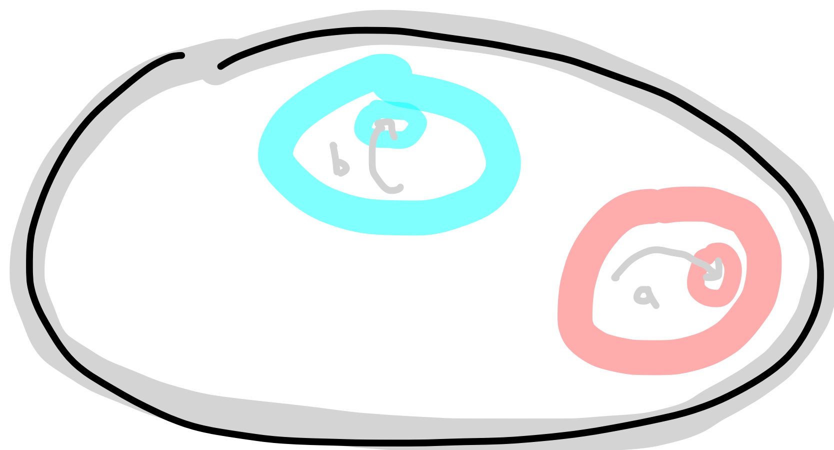
$t_1 = \pm i \dfrac{a+b}{a-b}$ $\qquad$ $t_1 = \pm i$ $\qquad$ $t_1 = \pm \left(- \dfrac{(c \pm d \pm e)^2 - (a+b)^2}{(c \pm d \pm e)^2 - (a-b)^2} \right)^{1/2}$

K3 automorphisms : Ping pong lemma

Problem: When is action $\mathbb{Z} * \mathbb{Z} \circlearrowright X$ faithful?

$= \langle a, b \rangle$

X:



Idea: Find disjoint subsets $U_a, U_b \subset X$ such that

- $a \cdot U_a \subset U_a$, $\quad b \cdot U_b = U_b$

- $a \cdot U_b \subset U_a \qquad b \cdot U_a \subset U_b$

K3 automorphisms : Ping pong lemma

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$\phantom{\mathbb{Z} * \mathbb{Z}} = \langle a, b \rangle$

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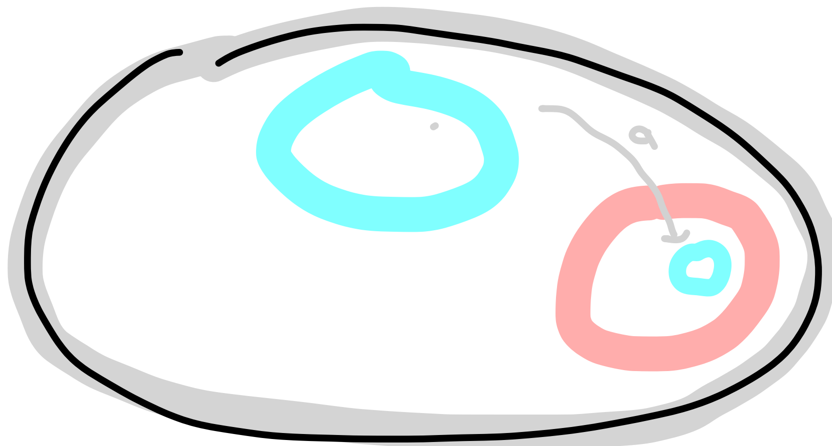

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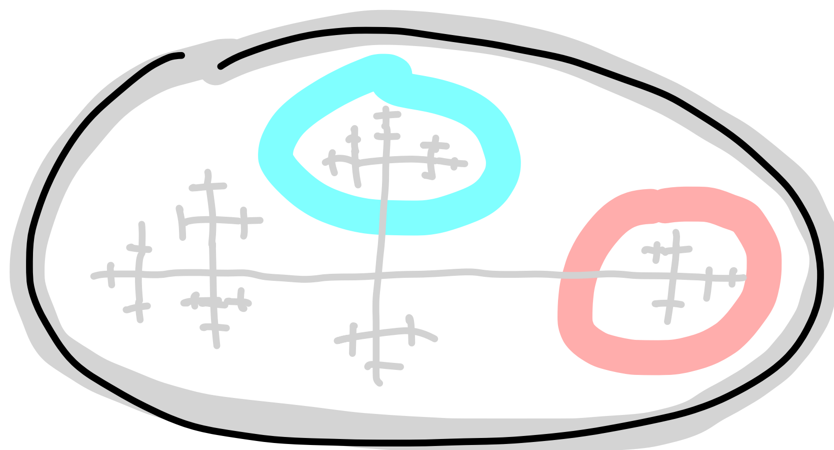

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K3 automorphisms of pentagon surfaces

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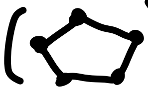
Coxeter() as subgroup [~~of finite index~~.]

infinite, hyperbolic

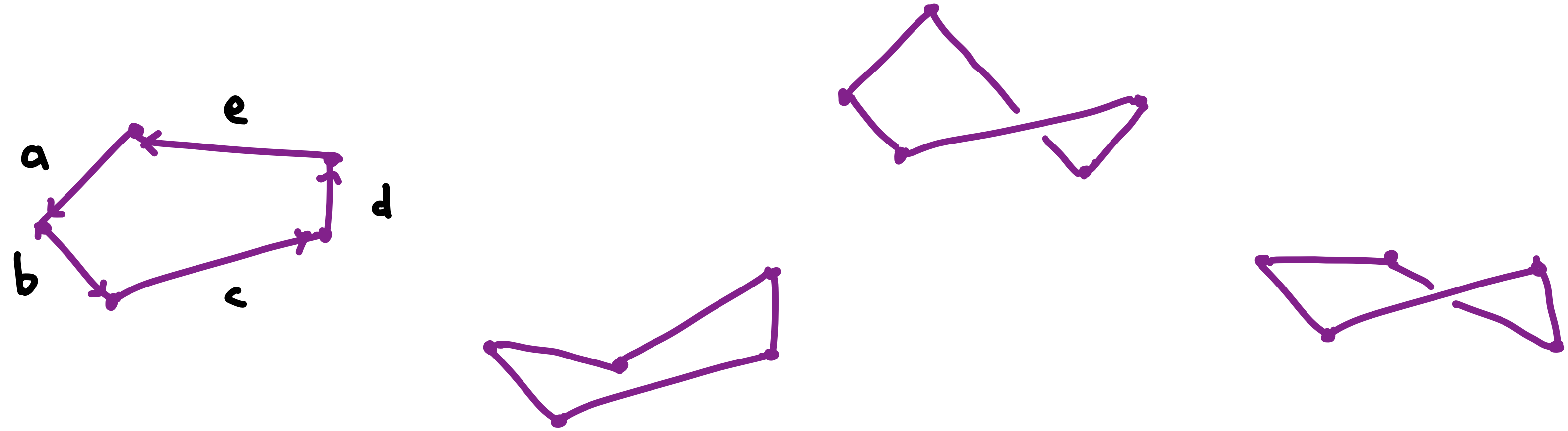
Proof idea :

- Use ping-pong lemma generalized to right-angled Coxeter groups (Genevois, 2019)

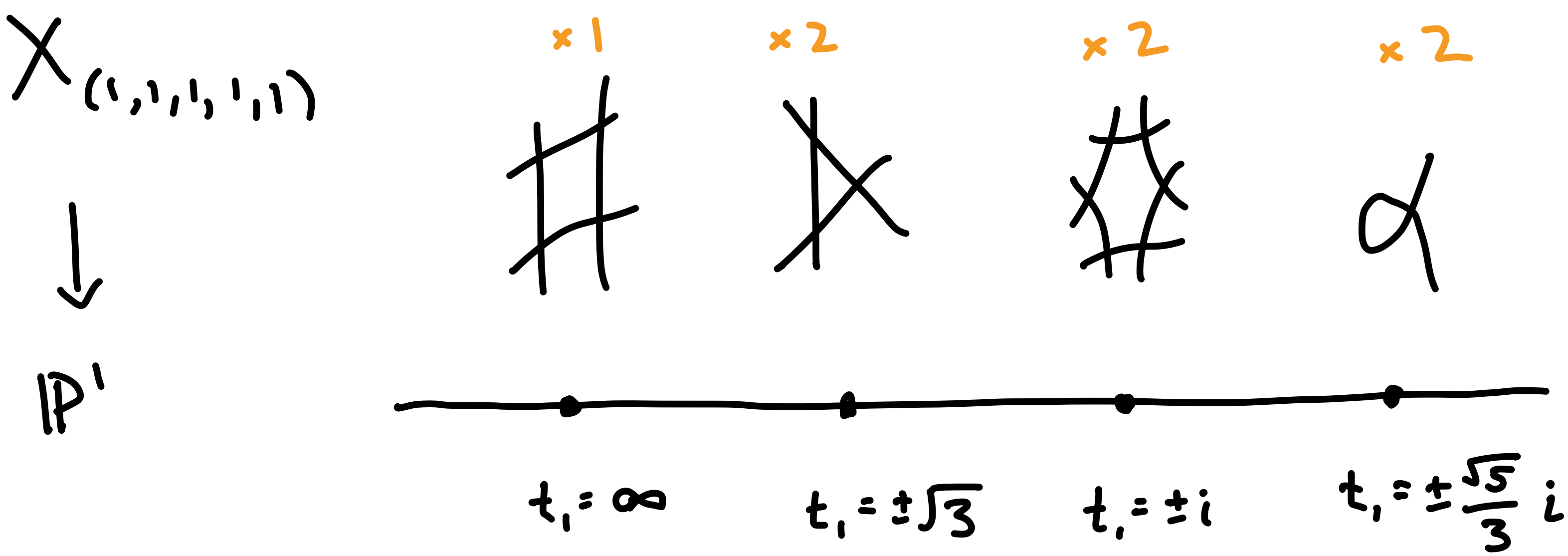
- For subsets U_i, use fact that every $X_{(a,\dots,e)}$ contains certain* complex points

$$(\pm i, \pm i, \dots, \pm i)$$

and consider tangent planes here


a
b
c
d
e

Thank you!


$X_{(1,1,1,1,1)}$
$\mathbb{P}^1$
×1
×2
×2
×2
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