The poset of floor quotients

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The **Möbius function** sends

$$\mu : \mathbb{N} \to \{-1, 0, 1\}$$

depending on the prime factorization.

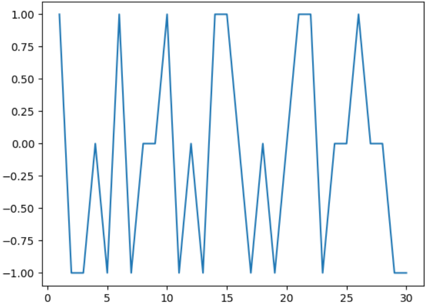

Figure: Graph of Möbius function

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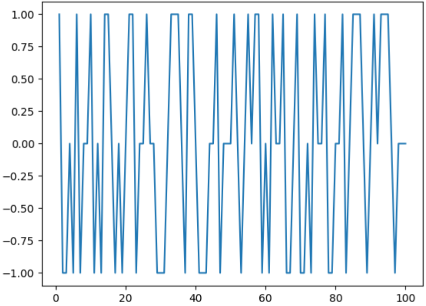
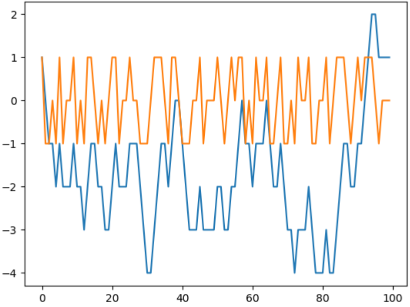


Figure: Graph of Möbius function

Mertens function

The **Mertens function** takes sums of the Möbius function

$$M(x) = \sum_{n \leq x} \mu(x)$$



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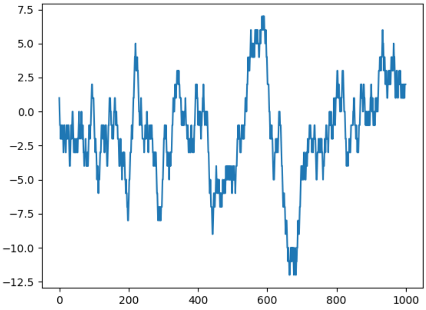

Figure: Graph of Mertens function

Why care?

Weighted prime counting $\Pi(x) := \sum_{p \leq x} \log p$

Theorem

$$R.\ H. \quad \Leftrightarrow \quad \Pi(x) = x + O(x^{1/2+\epsilon})$$

Mertens function $M(x) := \sum_{n \leq x} \mu(x)$ where $\mu(x)$ is Möbius function

Theorem

$$R.\ H. \quad \Leftrightarrow \quad M(x) = O(x^{1/2+\epsilon})$$

i.e. $\mu : \mathbb{N} \to \{-1, 0, 1\}$ "behaves like random"

Why care? Data

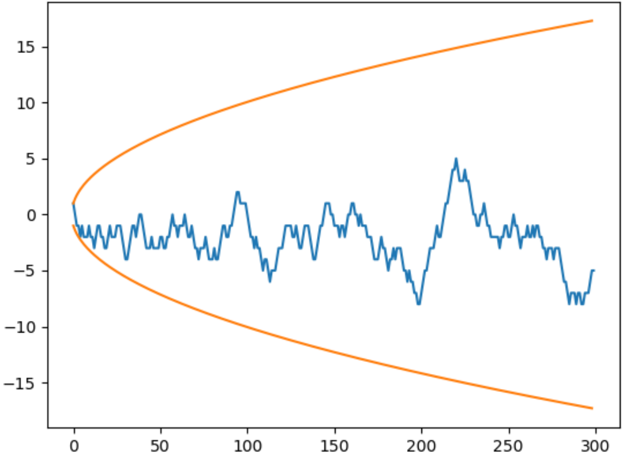

Figure: Mertens function

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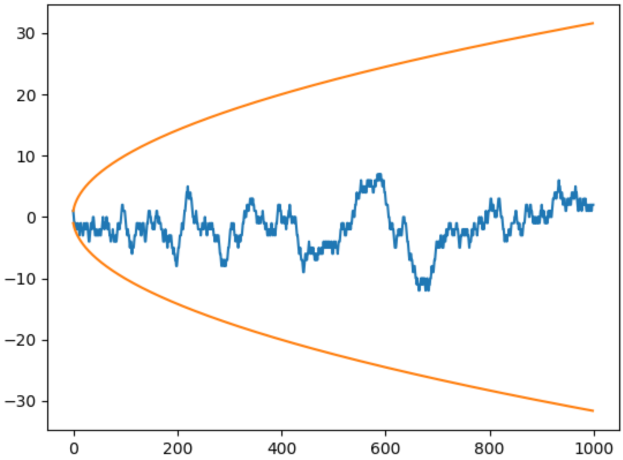

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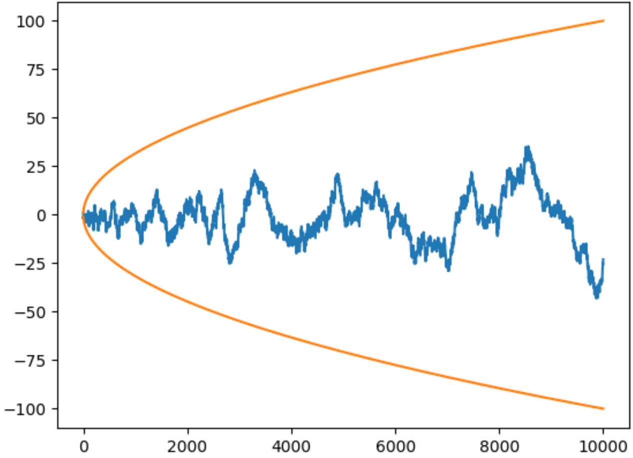

Figure: Mertens function

Almost divisors

The **divisors** of n are

$$\mathrm{Div}(n) = \left\{ d \in \mathbb{N} : d = \frac{n}{k} \text{ for some integer } k \right\}.$$

Ex. $\mathrm{Div}(16) = \{1, 2, 4, 8, 16\}$

The **floor quotients** (or "almost divisors") of n are

$$\mathrm{ADiv}(n) = \left\{ d \in \mathbb{N} : d = \left\lfloor \frac{n}{k} \right\rfloor \text{ for some integer } k \right\}.$$

Ex. $\mathrm{ADiv}(16) = \{1, 2, 3 = \left\lfloor \frac{16}{5} \right\rfloor, 4, 5 = \left\lfloor \frac{16}{3} \right\rfloor, 8, 16\}$

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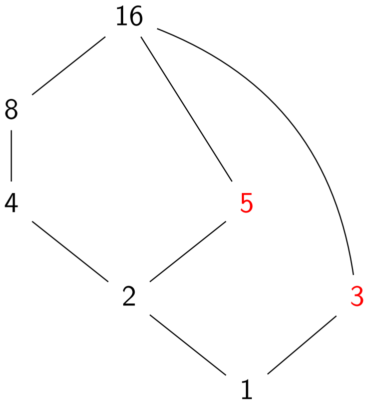
Theorem (Cardinal)

The floor quotient relation $d \preceq_{FQ} n$ defines a **partial order** *on $\mathbb{N}$.*

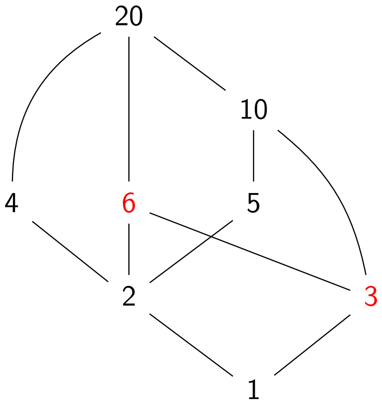
Floor quotient partial order

Ex. $n = 16$



Floor quotient partial order

Ex. $n = 20$

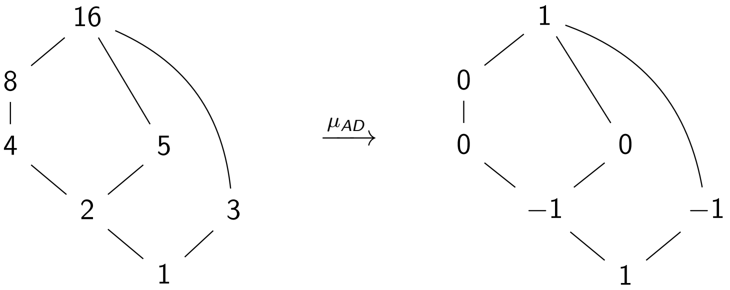


Floor quotient Möbius function

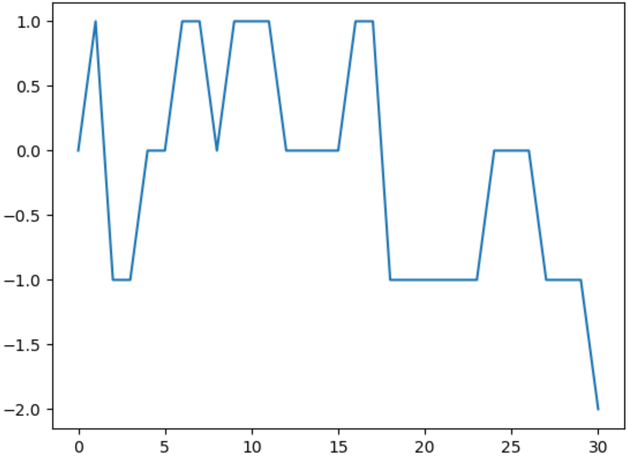
Möbius function of a partial order $(\mathbb{N}, \preceq_{FQ})$ is defined by

$$\mu_{FQ}(1) = 1, \qquad \mu_{FQ}(n) = - \sum_{\substack{d \preceq_{FQ} n \\ d \neq n}} \mu_{FQ}(d)$$

Ex. $n = 16$



Floor quotient Möbius function: Data



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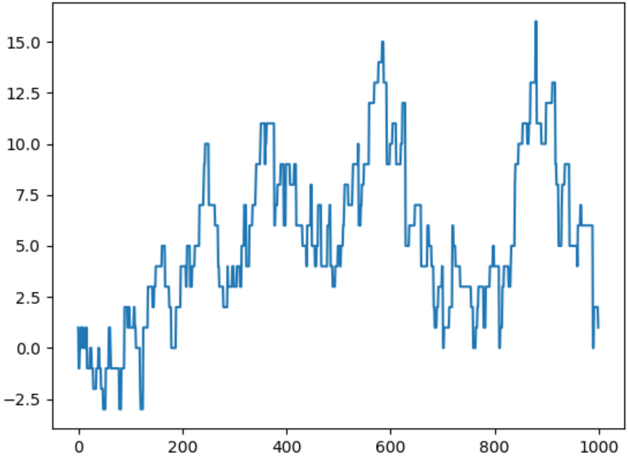


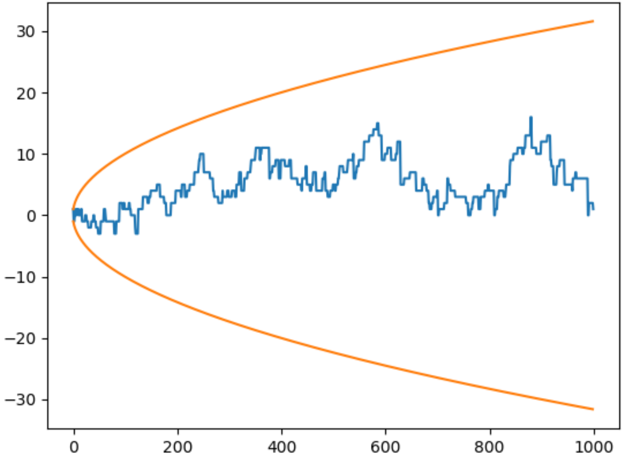
Bounding μ_{AD}

Problem

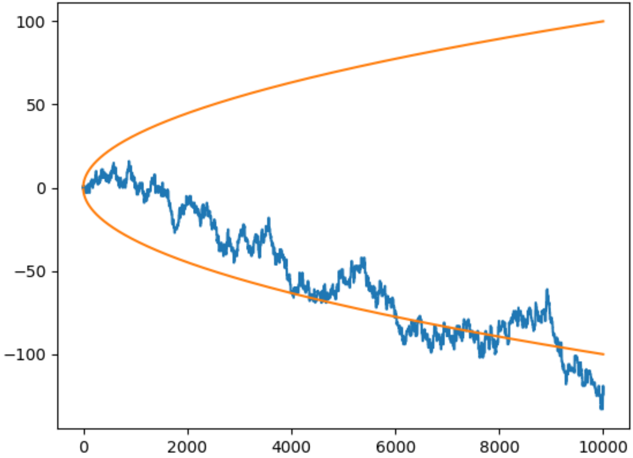
Does μ_{FQ} satisfy

$$\mu_{FQ}(n) = O(n^{1/2+\epsilon})?$$

Floor quotient Möbius function: Data



Floor quotient Möbius function: Data



Bounding μ_{AD}

Theorem (Lagarias–R)

The almost-divisor Möbius function μ_{AD} satisfies

$$\mu_{AD}(n) = O(n^{1.729}) \qquad \text{as} \quad n \to \infty.$$

The exponent satisifes $\zeta(1.729) < 2$

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Can we do any better?

Problem

Does μ_{FQ} satisfy

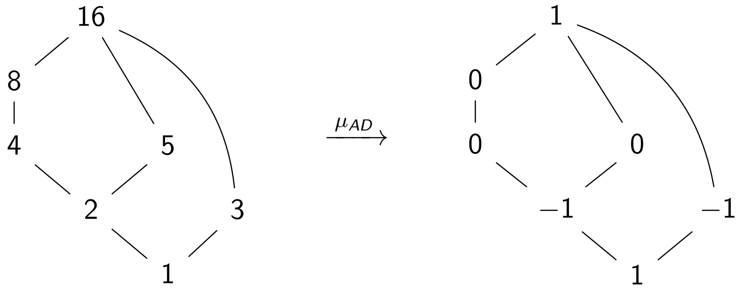
$$\mu_{FQ}(n) = O(n^{1/2+\epsilon})?$$

References

J.-P. Cardinal (2010)
Symmetric matrices related to the Mertens function
Lin. Alg. Appl. **432**(1), 161–172.

J. C. Lagarias and D. H. Richman
The floor quotient partial order
submitted.

Poset of floor quotients



Thank you!

Differenced floor quotient Möbius function

Observation: μ_{FQ} "behaves like" Mertens function

Let $\Delta\mu_{FQ}(n) = \mu_{FQ}(n) - \mu_{FQ}(n-1) \quad \Rightarrow$ "behaves like" usual Möbius

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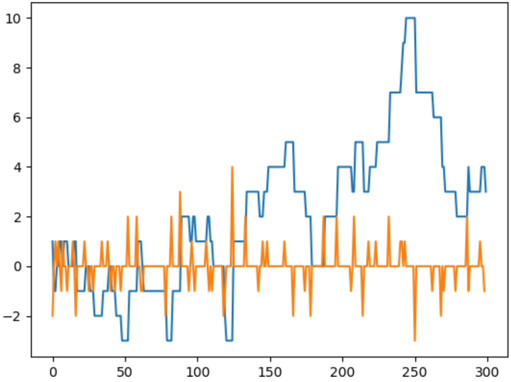

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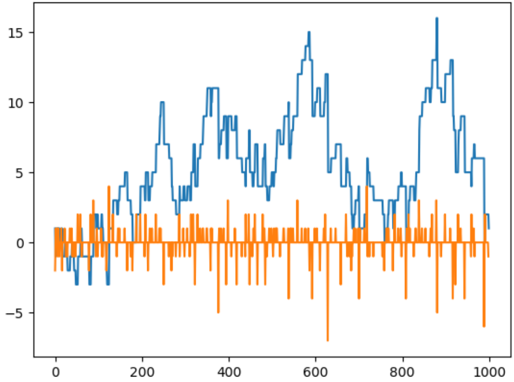

Differenced floor quotient Möbius function

Theorem (Lagarias–R)

The differenced almost-divisor Möbius function $\Delta\mu_{AD}$ for $n \geq 3$ satisfies the recursion

$$\Delta\mu_{AD}(n) = \begin{cases} -\sum\limits_{\substack{d|n \\ \sqrt{n}<d<n}} \Delta\mu_{FQ}(d) - \mu_{FQ}(s) & \text{if } n = s^2 \text{ or } s(s+1) \\ -\sum\limits_{\substack{d|n \\ \sqrt{n}<d<n}} \Delta\mu_{FQ}(d) & \text{otherwise.} \end{cases}$$

Recall that $\mu(n) = -\sum\limits_{\substack{d|n \\ 1\leq d<n}} \mu(d)$

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Differenced floor quotient Möbius function

Theorem (Lagarias–R)

$\Delta\mu_{FQ}(n) = 0$ if n satisfies either

1. *n is square-free and odd*
2. *n has prime divisor $p \geq \sqrt{n} + 1$*

Corollary

The density of the support of $\Delta\mu_{FQ}$ is

$$\lim_{x \to \infty} \frac{\#\{n \leq x : \Delta\mu_{FQ}(n) \neq 0\}}{x} < 0.183$$