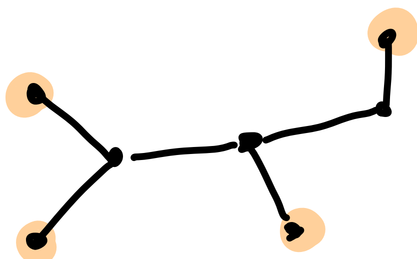
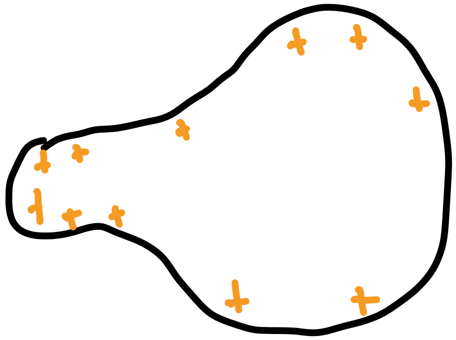
Tree distance matrices & their minors





Harry Richman

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24 June 2025

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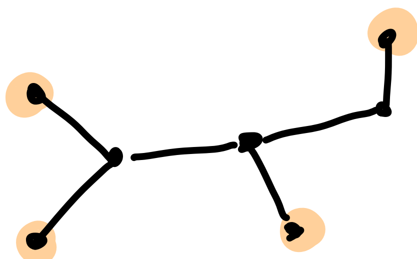
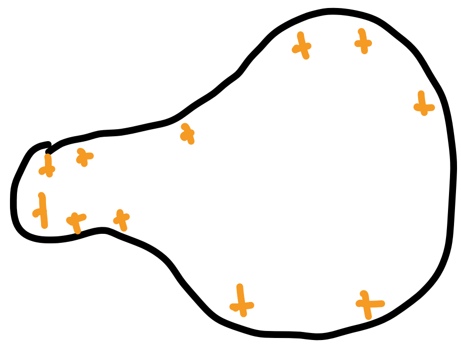
Tree distance matrices & their minors

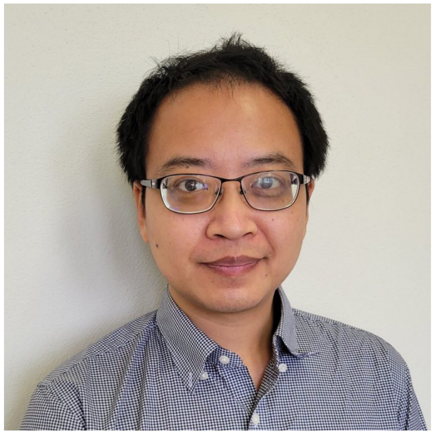




joint work w/

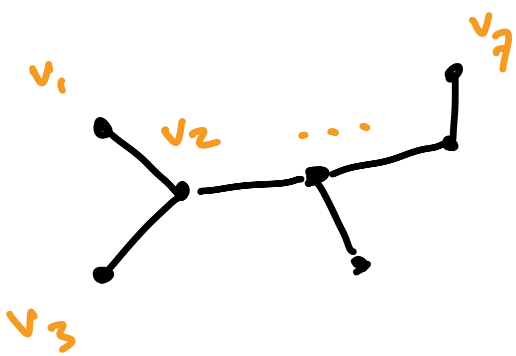
Farbod Shokrieh, Chenxi Wu

U. Washington U. Wisconsin



Distance matrices

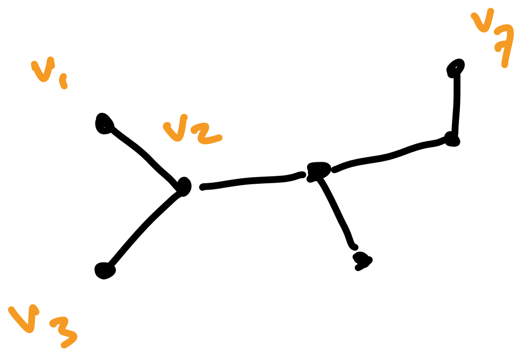
Problem What does determinant of a distance matrix tell us "combinatorially"?



$$D = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \\ \vdots \\ \\ v_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} v_1 & v_2 & v_3 & & \cdots & & v_7 \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 2 & 3 & 3 & 4 \\ 1 & 0 & 1 & 1 & 2 & 2 & 3 \\ 2 & 1 & 0 & 2 & 3 & 3 & 4 \\ 2 & 1 & 2 & 0 & 1 & 1 & 2 \\ 3 & 2 & 3 & 1 & 0 & 2 & 3 \\ 3 & 2 & 3 & 1 & 2 & 0 & 1 \\ 4 & 3 & 4 & 2 & 3 & 1 & 0 \end{bmatrix} \end{array}$$

Distance matrices

Problem What does the determinant tell us ?

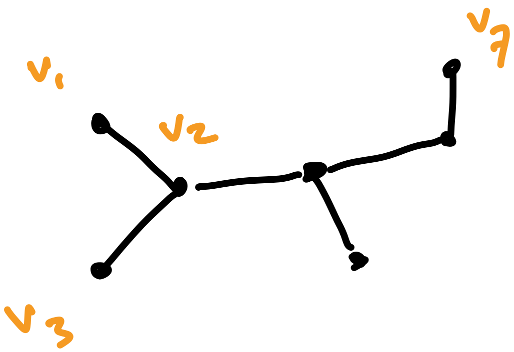


$$D = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \\ \vdots \\ \\ v_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} v_1 & v_2 & v_3 & & \cdots & & v_7 \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 2 & 3 & 3 & 4 \\ 1 & 0 & 1 & 1 & 2 & 2 & 3 \\ 2 & 1 & 0 & 2 & 3 & 3 & 4 \\ 2 & 1 & 2 & 0 & 1 & 1 & 2 \\ 3 & 2 & 3 & 1 & 0 & 2 & 3 \\ 3 & 2 & 3 & 1 & 2 & 0 & 1 \\ 4 & 3 & 4 & 2 & 3 & 1 & 0 \end{bmatrix} \end{array}$$

$$\Rightarrow \det D = 192$$

Distance matrices

Problem What does the determinant tell us?



$$D = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \\ \vdots \\ \\ v_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} v_1 & v_2 & v_3 & & \cdots & & v_7 \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 2 & 3 & 3 & 4 \\ 1 & 0 & 1 & 1 & 2 & 2 & 3 \\ 2 & 1 & 0 & 2 & 3 & 3 & 4 \\ 2 & 1 & 2 & 0 & 1 & 1 & 2 \\ 3 & 2 & 3 & 1 & 0 & 2 & 3 \\ 3 & 2 & 3 & 1 & 2 & 0 & 1 \\ 4 & 3 & 4 & 2 & 3 & 1 & 0 \end{bmatrix} \end{array}$$

Theorem (Graham-Pollak, 1971)

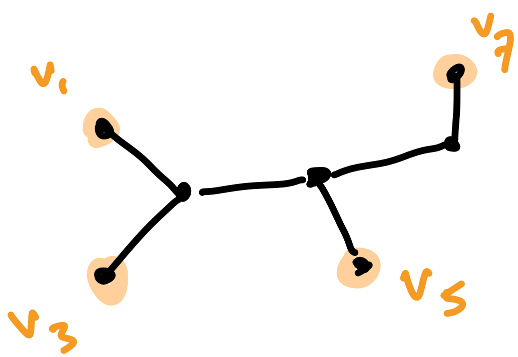
For any tree on n vertices,

$$\det D = (-1)^{n-1} 2^{n-2} (n-1)$$

no combinatorics :(

Distance matrices

Problem What does determinant of a distance submatrix tell us "combinatorially"?



$$D = \begin{bmatrix} 0 & 1 & 2 & 2 & 3 & 3 & 4 \\ 1 & 0 & 1 & 1 & 2 & 2 & 3 \\ 2 & 1 & 0 & 2 & 3 & 3 & 4 \\ 2 & 1 & 2 & 0 & 1 & 1 & 2 \\ 3 & 2 & 3 & 1 & 0 & 2 & 3 \\ 3 & 2 & 3 & 1 & 2 & 0 & 1 \\ 4 & 3 & 4 & 2 & 3 & 1 & 0 \end{bmatrix}$$

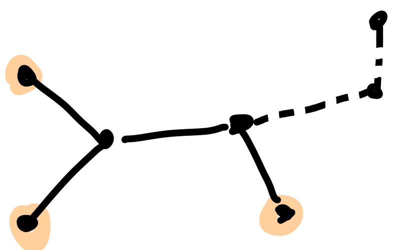
$$D[S] = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 0 & 3 & 4 \\ 3 & 3 & 0 & 3 \\ 4 & 4 & 3 & 0 \end{bmatrix}$$

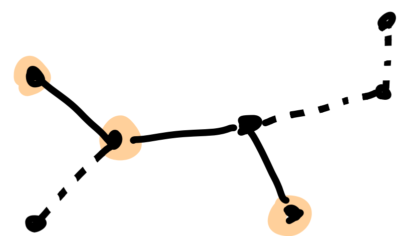
e.g.

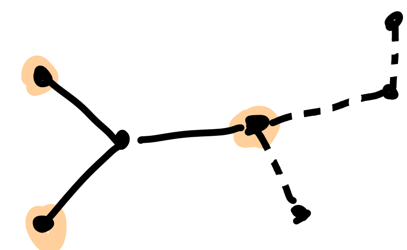
$S = \partial G$

Distance matrices

Problem: What does determinant of a distance submatrix tell us "combinatorially"?



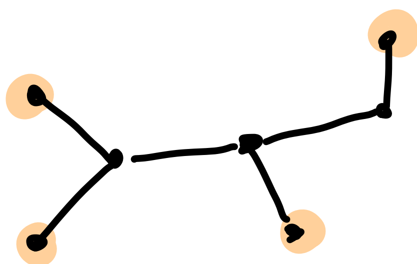




$$\begin{bmatrix} 0 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$$

Distance matrices

Problem What does determinant tell us ?



$$D[S] = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 0 & 3 & 4 \\ 3 & 3 & 0 & 3 \\ 4 & 4 & 3 & 0 \end{bmatrix}$$

$\Rightarrow$ $\det D[S] = -252$??

⚠ No previously known combinatorial interpretation

Distance matrices

Thm (Graham-Pollak)

$$\det D = (-1)^{n-1} 2^{n-2} (n-1)$$

Theorem (R-Shokrieh-Wu)

Given a tree $G=(V,E)$ and vertex subset $S \subset V$,

$$\det D[S] = (-1)^{|S|-1} 2^{|S|-2} \left((n-1)\, \kappa_1(G;S) - \sum_{F \in \mathcal{F}_2(G;S)} \left(\deg^{\circ}(F,*)-2\right)^2 \right)$$

where n = # vertices

$\kappa_1(G;S)$ = # S-rooted spanning forests

$\mathcal{F}_2(G;S)$ = $(S,*)$-rooted spanning forests

$\deg^{\circ}(F,*)$ = out-degree of floating component

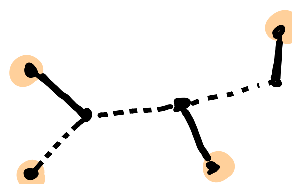
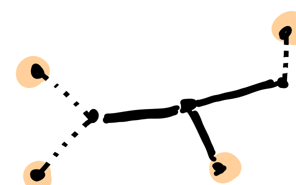
Distance matrices

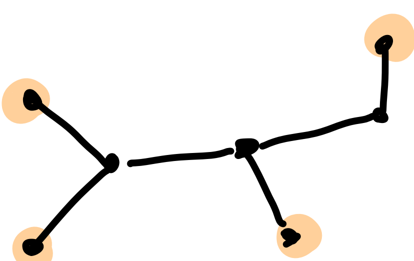
Theorem (R - Shokrieh - Wu)

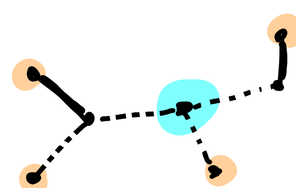
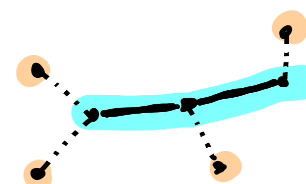
Given a tree $G = (V, E)$ and vertex subset $S \subset V$,

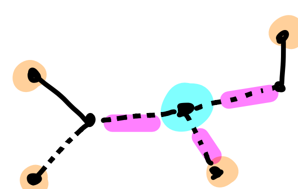
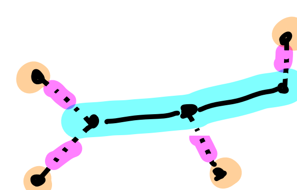
$$\det D[S] = (-1)^{|S|-1} \, 2^{|S|-2} \left((n-1)\, \kappa_1(G;S) - \sum_{F \in \mathcal{F}_2(G;S)} \left(\deg^\circ(F,*)-2\right)^2 \right)$$

Ex. where $n = 7$

$\kappa_1(G;S) = \#\{$  ,  , $\dots \}$



$\mathcal{F}_2(G;S) = \{$  ,  , $\dots \}$

$\deg^\circ(F,*) =$  $\deg^\circ = 3$  $\deg^\circ = 4$

Distance matrices

<u>Theorem</u> (R - Shokrieh - Wu)

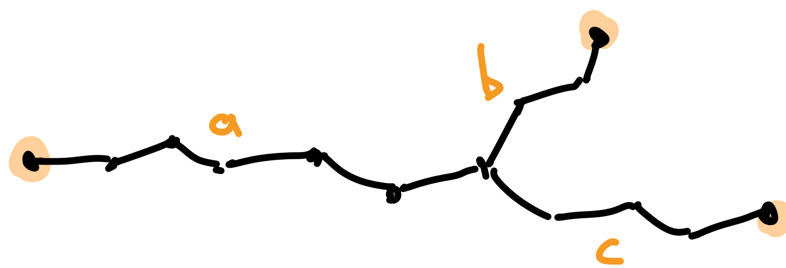
$$\det D[S] = (-1)^{|S|-1}\, 2^{|S|-2} \left((n-1)\, \kappa_1(G;S) - \sum_{F \in \mathcal{F}_2(G;S)} \left(\deg^{o}(F,*) - 2\right)^2 \right)$$

How to prove?

 Potential theory on trees

Distance matrices

Ex. Tripod graph



$$D[S] = \begin{bmatrix} 0 & a+b & a+c \\ a+b & 0 & b+c \\ a+c & b+c & 0 \end{bmatrix}$$

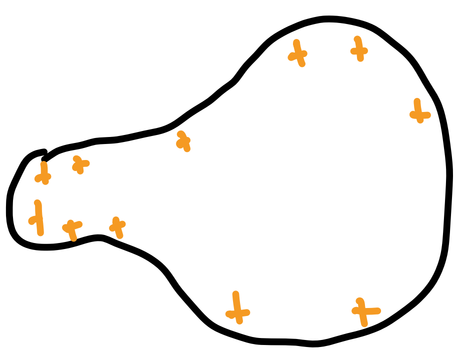
$$\det D[S] = 2(a+b)(a+c)(b+c)$$

By Theorem,

$$\det D[S] = 2\Big(\underbrace{(a+b+c)}_{n-1}\underbrace{(ab+bc+ac)}_{\kappa_1(G;S)} - \underbrace{abc}_{\substack{\sum 1^2 \\ \mathcal{F}_2(G;S)}}\Big)$$

Potential theory

Problem How do particles "distribute" within a region, given repulsive potential $U(x,y)$?

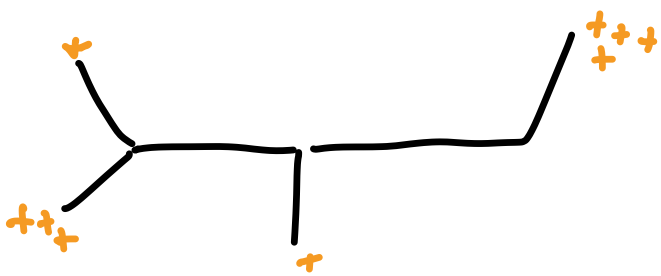


2-dim region

1-dim tree

Potential theory: tree case

Problem How do particles distribute?



1-dim tree

- Minimize

$$\mathcal{E}(\vec{\mu}) = -\frac{1}{2}\,\vec{\mu}^T D[S]\,\vec{\mu}$$

→ self-repulsion energy

- Constraint

$$\vec{\mathbb{1}} \cdot \vec{\mu} = 1$$

→ conservation of mass

Potential theory

Problem Find

$$\min \left\{ -\frac{1}{2} \vec{\mu}^T D[S] \vec{\mu} \ : \ \vec{\mu} \in \mathbb{R}^S, \ \mathbb{1} \cdot \vec{\mu} = 1 \right\}$$

$\nabla(\text{objective}) = -D[S]\vec{\mu}$ $\qquad$ $\nabla(\text{constraint}) = \mathbb{1}$

Proposition (c.f. Bapat)

a) Minimum occurs at $D[S]\vec{\mu}^* = \lambda \mathbb{1}$

b) $$\min_{\mathbb{1}\cdot\vec{\mu}=1} \left\{ -\frac{1}{2} \vec{\mu}^T D[S] \vec{\mu} \right\} = -\frac{1}{2} \frac{\det D[S]}{\text{cof } D[S]}$$

sum of cofactors $\sum_{i,j} (-1)^{i+j} \det A_{i,j}$

Potential theory

$$\mathcal{E} = -\frac{1}{2} \vec{\mu}^T D[S] \vec{\mu}$$

<u>Proposition</u> (Bapat)

a) Minimum $\mathcal{E}(\vec{\mu})$ occurs at $D[S] \vec{\mu}^* = \lambda \mathbb{1}$

b) $$\min_{\mathbb{1} \cdot \vec{\mu} = 1} \left\{ \mathcal{E}(\vec{\mu}) \right\} = -\frac{1}{2} \frac{\det D[S]}{\operatorname{cof} D[S]}$$

sum of cofactors $\sum_{i,j} (-1)^{i+j} \det A_{i;j}$

Aside

<u>Theorem</u> (RSW + Bapat - Sivasubramanian 2011)

$$\frac{\det D[S]}{\operatorname{cof} D[S]} = \frac{1}{2} \left((n-1) - \frac{\sum_{F \in \mathcal{F}_2(G;S)} (\deg^o(F,*) - 2)^2}{\kappa_1(G;S)} \right)$$

Potential theory

$$\mathcal{E} = -\frac{1}{2}\vec{\mu}^T D[S]\vec{\mu}$$

<u>Proposition</u> (Bapat)

a) Minimum $\mathcal{E}(\vec{\mu})$ occurs at $D[S]\,\vec{\mu}^* = \lambda \mathbb{1}$

b) $$\min_{\mathbb{1}\cdot\vec{\mu}=1} \left\{ \mathcal{E}(\vec{\mu}) \right\} = -\frac{1}{2}\,\frac{\det D[S]}{\operatorname{cof} D[S]}$$

sum of cofactors $\sum_{i,j} (-1)^{i+j} \det A_{i,j}$

Aside

<u>Theorem</u> (RSW, also Devriendt 2022)

If $A \subset B \subset V(G)$, then $$\frac{\det D[A]}{\operatorname{cof} D[A]} \le \frac{\det D[B]}{\operatorname{cof} D[B]}$$

Potential theory

Summary:

How to find $\det D[S]$? $\rightsquigarrow$ How to find $\min \mathcal{E}(\vec{\mu})$? $\rightsquigarrow$ How to solve $D[S]\vec{\mu} = \lambda \mathbb{1}$?

Theorem (Bapat - Sivasubramanian 2011, et al. ?)

Equilibrium vector is

$$\begin{cases} D[S]\vec{\mu}^* = \lambda \mathbb{1} \\ \mathbb{1} \cdot \vec{\mu}^* = 1 \end{cases}$$

$$\mu_i^* = \frac{1}{2\, \kappa_1(G;S)} \sum_{T \in \mathcal{F}_1(G;S)} \left(2 - \deg^\circ(T, i) \right)$$

Potential theory

Theorem (Bapat – Sivasubramanian 2011)

$$\mu_i^* = \frac{1}{2\,\kappa_i(G;S)} \sum_{T \in \mathcal{F}_i(G;S)} \left(2 - \deg^\circ(T,i)\right)$$

Idea: $\boxed{\text{combinatorics of } D}$ $\longleftrightarrow$ $\boxed{\text{combinatorics of Laplacian}}$

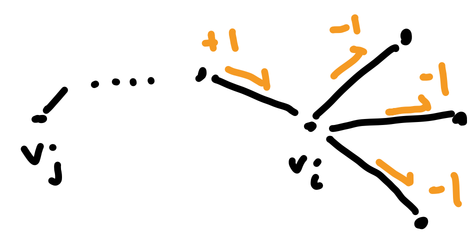
- Laplacian $\qquad (L\vec{x})_i = \sum_{j \sim i} (x_i - x_j)$

- L times distance matrix

$$(LD)_{i,j} = \left(L\begin{bmatrix} | \\ d_{\cdot j} \\ | \end{bmatrix}\right)_i = 1 - (\deg_i - 1) = 2 - \deg_i$$

no j-index

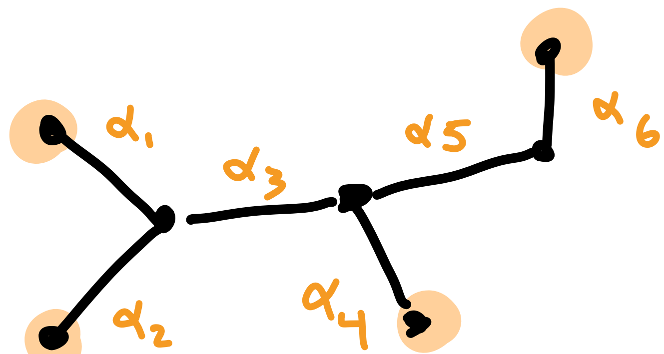
G:



except when $i = j$

Further extensions: edge weights

Ex.



$$D = \begin{bmatrix} 0 & \alpha_1 & \alpha_1 + \alpha_2 & \cdots \\ \alpha_1 & 0 & \alpha_2 & \\ \alpha_1 + \alpha_2 & \alpha_2 & 0 & \\ \vdots & \vdots & & \ddots \end{bmatrix}$$

edge weights

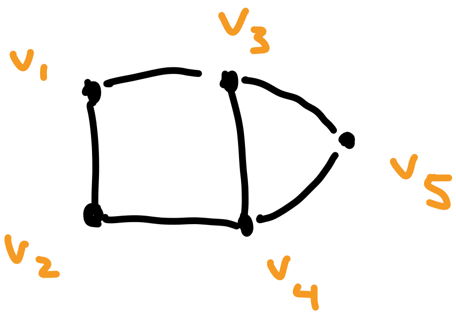
Theorem (R - Shokrieh - Wu)

$$\det D[S] = (-1)^{|S|-1}\, 2^{|S|-2} \left(\sum_E \alpha_e \sum_{T \in \mathcal{F}_1} w(T) - \sum_{F \in \mathcal{F}_2} \left(\deg^{\circ}(F,*) - 2\right)^2 w(F) \right)$$

(Bapat - Kirkland - Neumann, 2005 when $S = V$)

Further extensions: graphs w/ cycles

Ex.

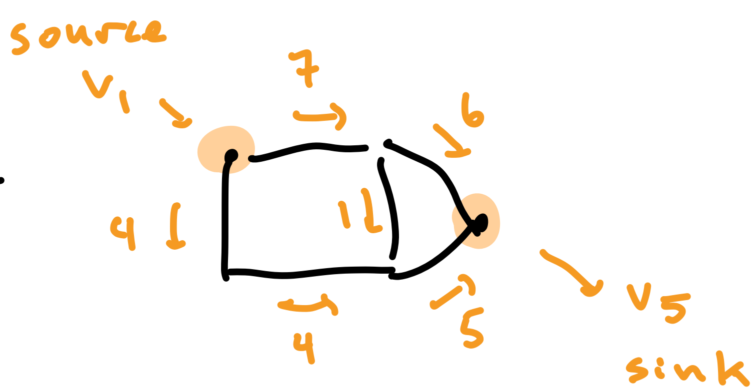


$$\cancel{D} = \begin{bmatrix} 0 & 1 & 1 & 2 & 2 \\ 1 & 0 & 2 & 1 & 2 \\ 1 & 2 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 1 \\ 2 & 2 & 1 & 1 & 0 \end{bmatrix}$$

effective resistance matrix

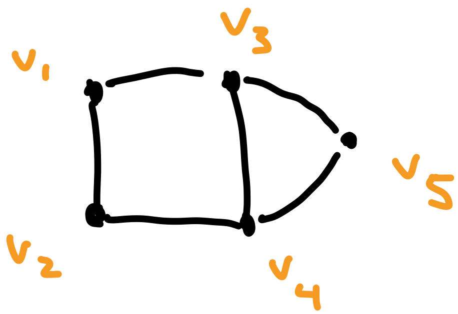
$$R = \begin{bmatrix} 0 & 8/11 & 8/11 & 10/11 & 13/11 \\ 8/11 & 0 & 10/11 & 8/11 & 13/11 \\ 8/11 & 10/11 & 0 & 6/11 & 7/11 \\ 10/11 & 8/11 & 6/11 & 0 & 7/11 \\ 13/11 & 13/11 & 7/11 & 7/11 & 0 \end{bmatrix}$$

$R_{1,5}$



Further extensions: graphs w/ cycles

Ex.



$$R = \begin{bmatrix} 0 & 8/11 & 8/11 & 10/11 & 13/11 \\ 8/11 & 0 & 10/11 & 8/11 & 13/11 \\ 8/11 & 10/11 & 0 & 6/11 & 7/11 \\ 10/11 & 8/11 & 6/11 & 0 & 7/11 \\ 13/11 & 13/11 & 7/11 & 7/11 & 0 \end{bmatrix}$$

Theorem (RSW)

$$\frac{\det R}{\operatorname{cof} R} = \frac{2}{3} \frac{\overbrace{\kappa_2(G)}^{\#\,2\text{-forests}}}{\underbrace{\kappa(G)}_{\#\text{ trees}}} - \frac{1}{6} \sum_{e \in E} \left(\frac{\kappa(G/e)}{\kappa(G)} \right)^2$$

Further extensions

Still unresolved:

- q-distance matrices, e.g. $D_q[S] = \frac{1}{(1-q)^4}\begin{bmatrix} 0 & 1-q^2 & \cdots \\ 1-q^2 & 0 & \\ 1-q^3 & 1-q^3 & \\ 1-q^4 & 1-q^4 & \cdots \end{bmatrix}$

  Bapat - Lai - Pati 2006

  Choudhury - Khare 2024

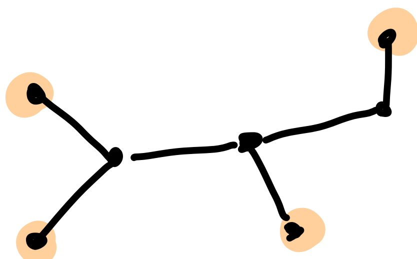
- Steiner distance hyperdeterminants, k-subsets

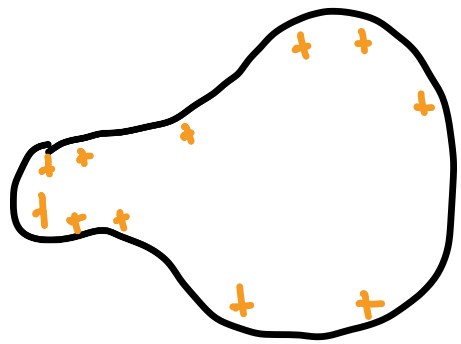
  Cooper - Tauscheck 2024+

- Combinatorial proof via sign-reversing involution

  Briand - Esquivias - Gutiérrez - Lillo - Rosas 2024+

Thank you!





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24 June 2025

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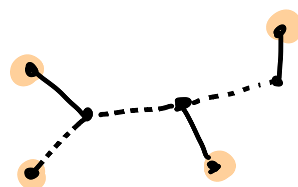
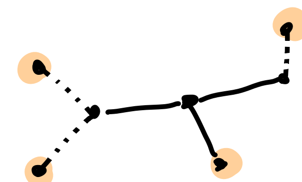
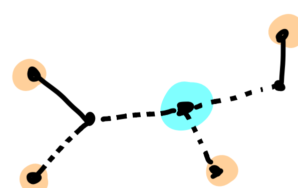
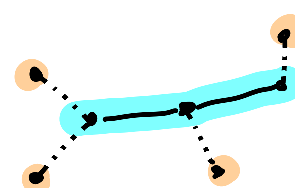
Aside: Transitions between $\mathcal{F}_1(G;S)$ and $\mathcal{F}_2(G;S)$ form interesting dynamical system

$\mathcal{F}_1(G;S) = \{$  ,  $\}$

delete edge $e \in T$

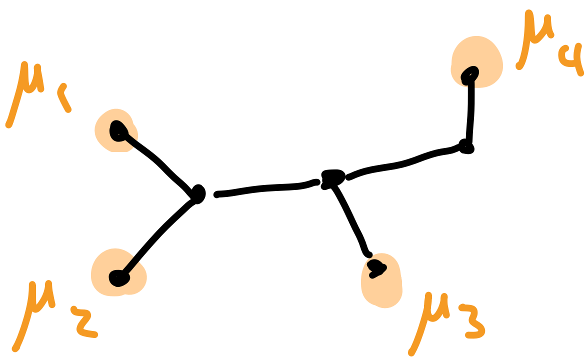
add edge $e \in \partial F(*)$

$\mathcal{F}_2(G;S) = \{$  ,  $\}$

see: Amini et al., Brandén-Huh, Vinzant et al

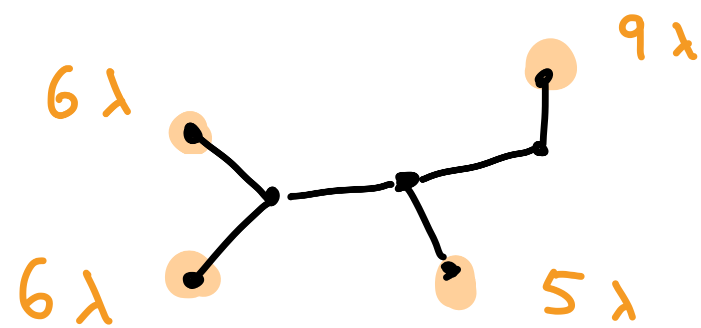
Potential theory

Ex.



$\rightsquigarrow$

Equilibrium



equilibrium ratio

$$D[S]\vec{\mu}^* = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 0 & 3 & 4 \\ 3 & 3 & 0 & 3 \\ 4 & 4 & 3 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 6 \\ 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 63 \\ 63 \\ 63 \\ 63 \end{bmatrix}$$

Recall:

$$\det D[S] = -252$$

$$= -4(63)$$