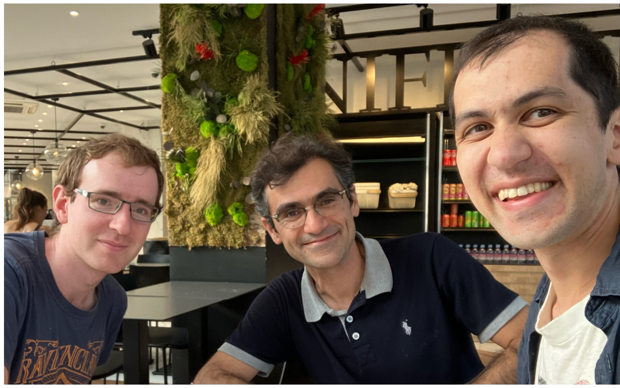
Weights of tropical Weierstrass points

14 September 2024

AMS Sectional, San Antonio

Harry Richman, Fred Hutch / NCTS
(Seattle) (Taipei)

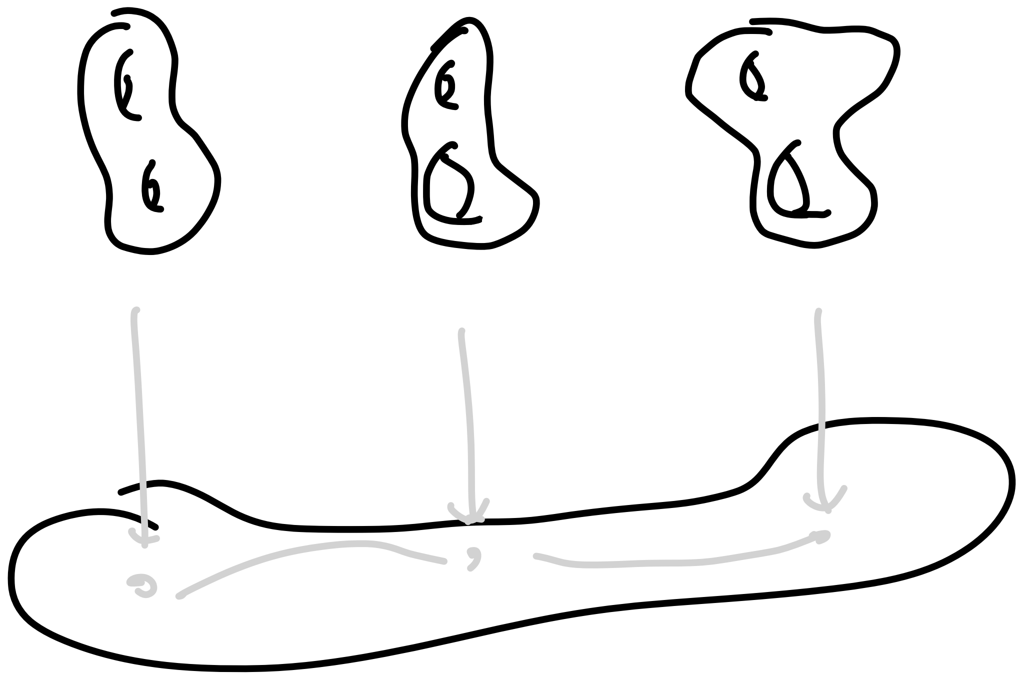


joint w/ Omid Amini & Lucas Gierczak Ecolé Polytechnique

Why Weierstrass points?

Goal: understand moduli space of algebraic curves $\mathcal{M}_g$

$$\begin{array}{c} \mathcal{C}_g \\ \downarrow \\ \mathcal{M}_g \end{array}$$

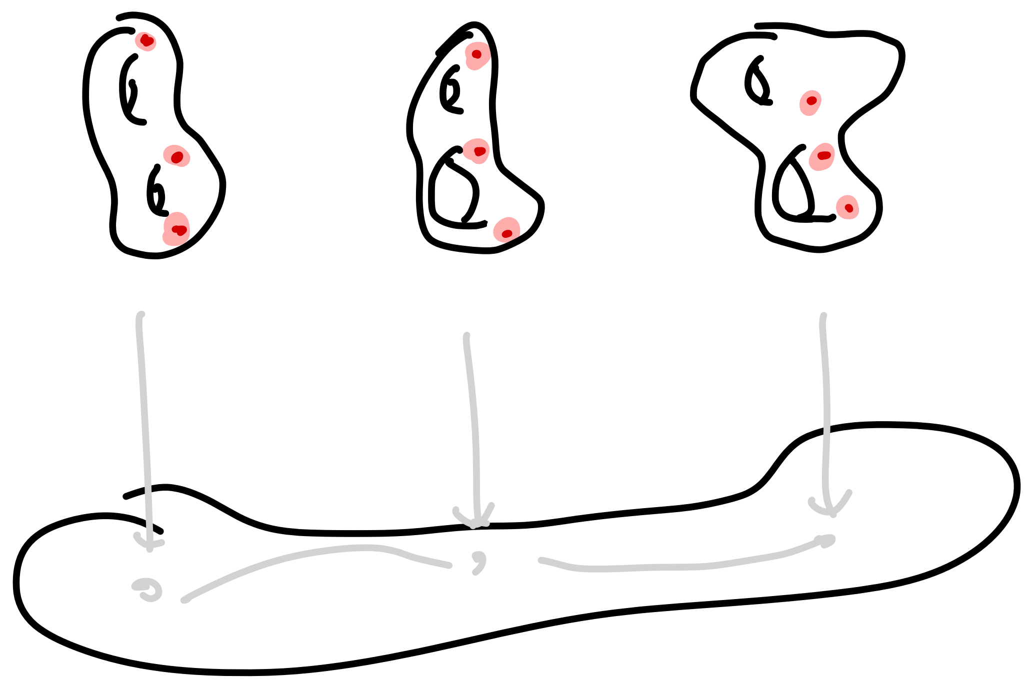


$(g = 2)$

Why Weierstrass points?

Goal: understand moduli space of algebraic curves $\mathcal{M}_g$

$$\begin{array}{c} C_g \\ \downarrow \\ \mathcal{M}_g \end{array}$$

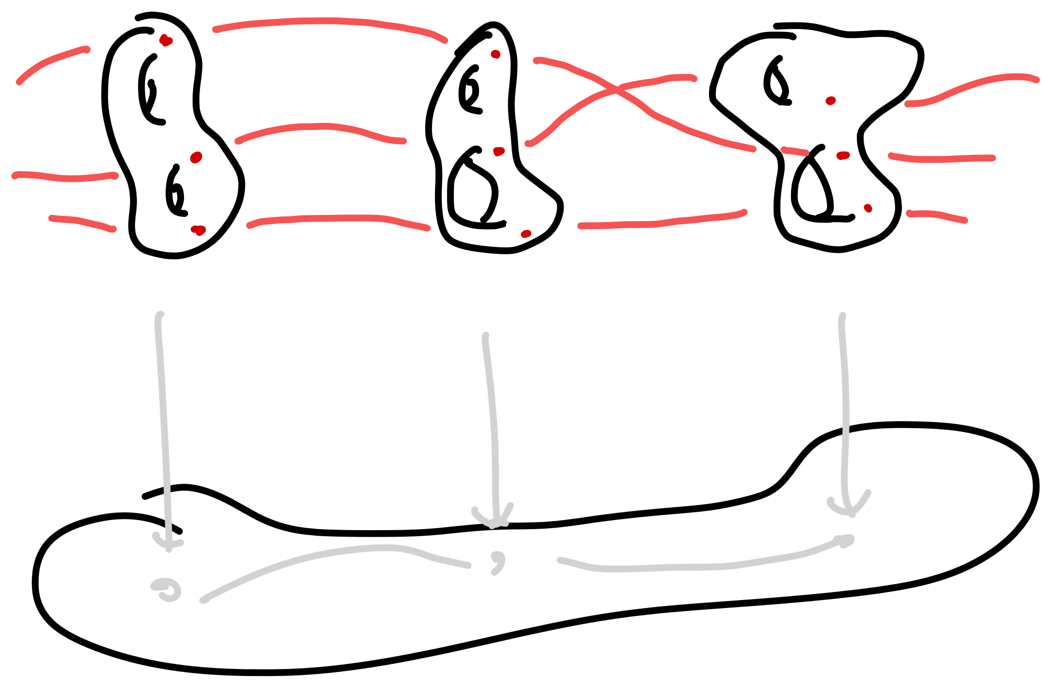


Slogan: "Weierstrass pts are coordinates on $\mathcal{M}_g$"

Why Weierstrass points?

Goal: understand moduli space of algebraic curves $\mathcal{M}_g$

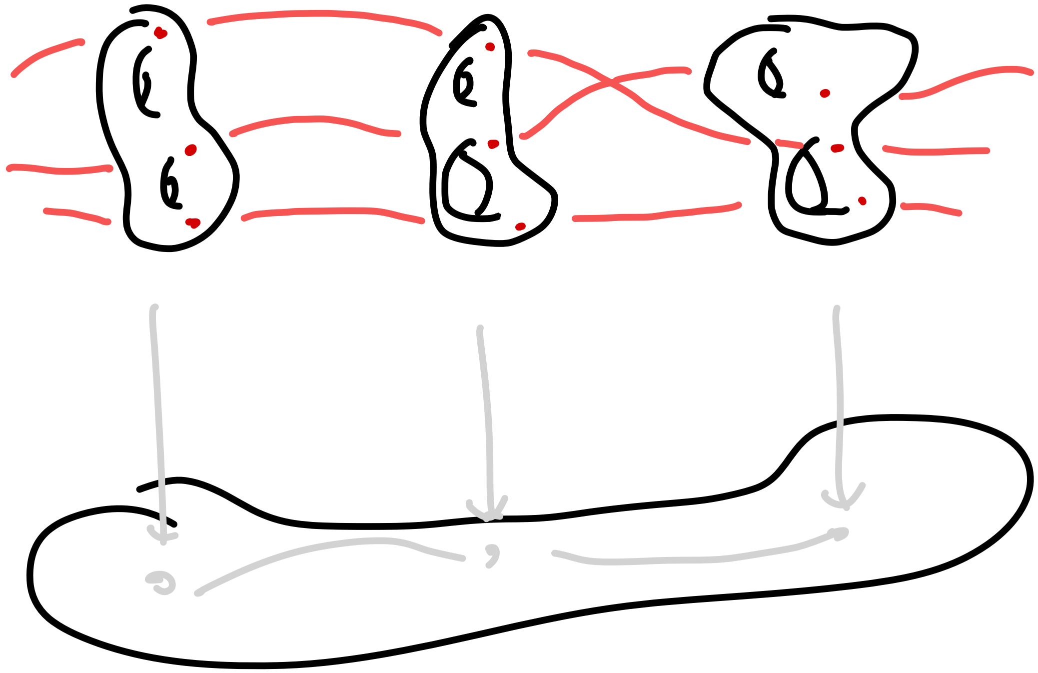
$$\begin{array}{c} \mathcal{C}_g \\ \downarrow \\ \mathcal{M}_g \end{array}$$



Slogan: "Weierstrass pts are coordinates on $\mathcal{M}_g$"

Why Weierstrass points?

$$C_g \downarrow \overline{M}_g \supset M_g$$

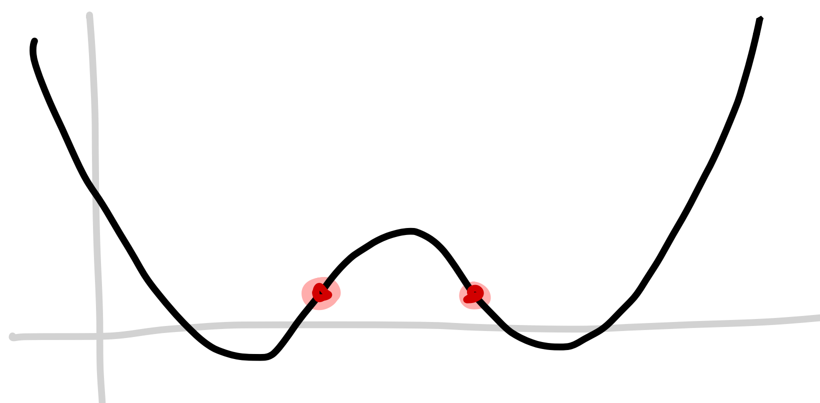


Problem: How do Weierstrass points behave on boundary of $\overline{M}_g$?

What are Weierstrass points? (Background)

Recall: first-year calculus, inflection point of a graph

Ex.



$y = x^4 - x^3 + \ldots$

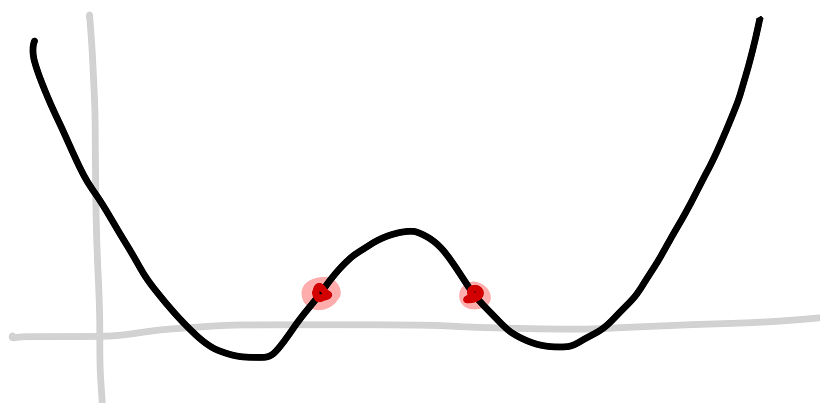
Inflection?

- "concavity changes sign"
- tangent line intersects graph w/ multiplicity ≥ 3

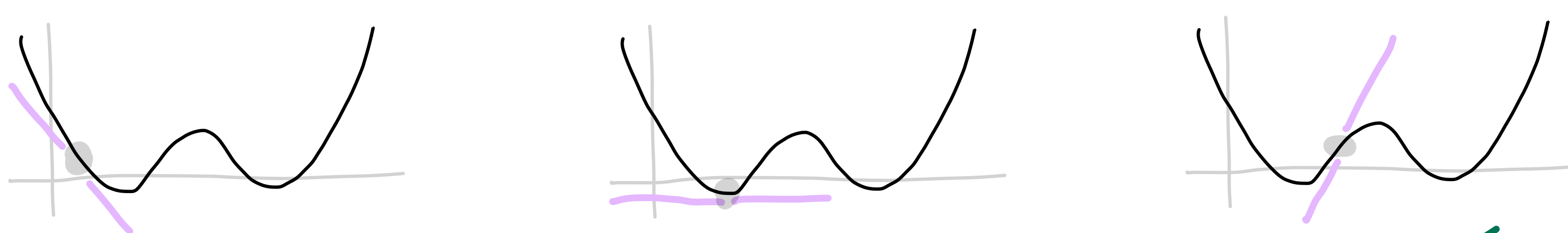
What are Weierstrass points? (Background)

Recall: first-year calculus, inflection point of a graph

Ex.



$y = x^4 - x^3 + \dots$

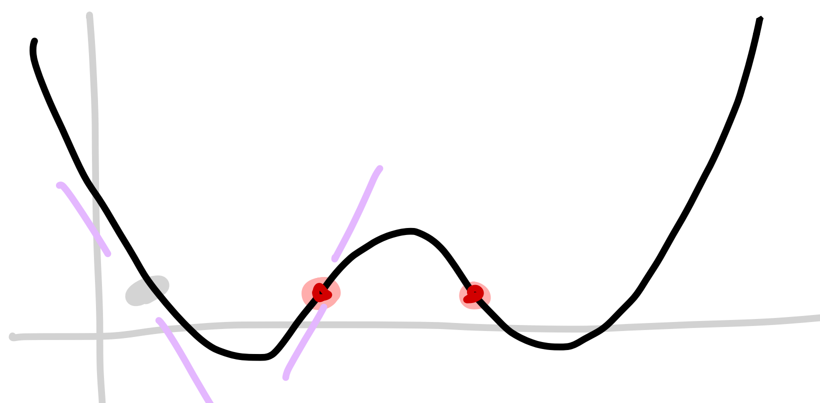


Inflection? ✗ ✗ ✓

What are Weierstrass points?

Recall: first-year calculus, inflection point of a graph

Ex.



$y = x^4 - x^3 + \cdots$

Fact: If polynomial $f(x)$ has degree d,

$y = f(x)$ has $d-2^*$ inflection pts

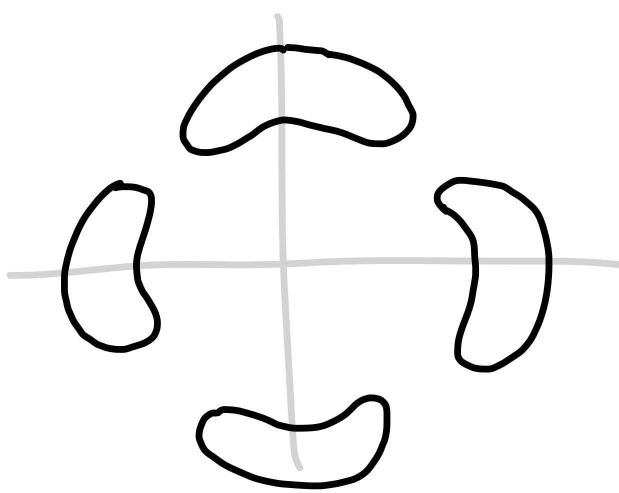
* counted w/ multiplicity

What are Weierstrass points?

X an alg. curve of genus $g \geq 2$

Def'n. Weierstrass points of X are inflection pts of canonical embedding $X \to \mathbb{P}^{g-1}$

Ex.



$g = 3$

$$144(x^4+y^4) - 225(x^2+y^2) + 350x^2y^2 + 81 = 0$$

"Trott curve" via Wikipedia

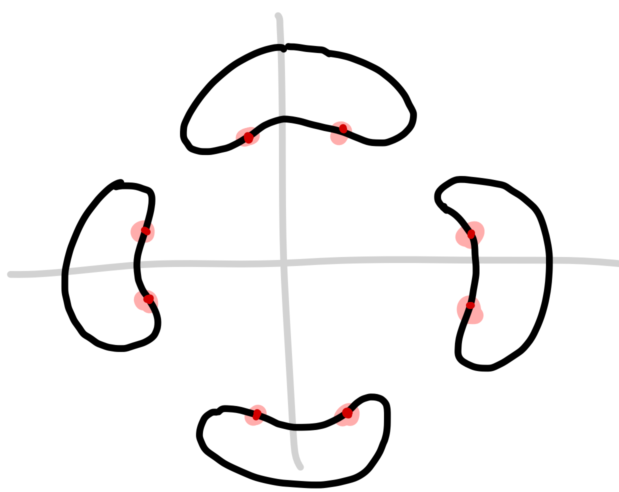
Q: How many inflection pts?

What are Weierstrass points?

X an alg. curve of genus $g \geq 2$

Def'n. Weierstrass points of X are inflection pts of canonical embedding $X \to \mathbb{P}^{g-1}$

Ex.



$$144(x^4+y^4) - 225(x^2+y^2) + 350x^2y^2 + 81 = 0$$

Fact: generic alg. curve has $g^3 - g$ distinct Weierstrass pts

What are Weierstrass points?

X an alg. curve of genus $g \geq 2$

Def'n. Weierstrass points of X are inflection pts of canonical embedding $X \to \mathbb{P}^{g-1}$

$\Big\Updownarrow$ Divisor theory

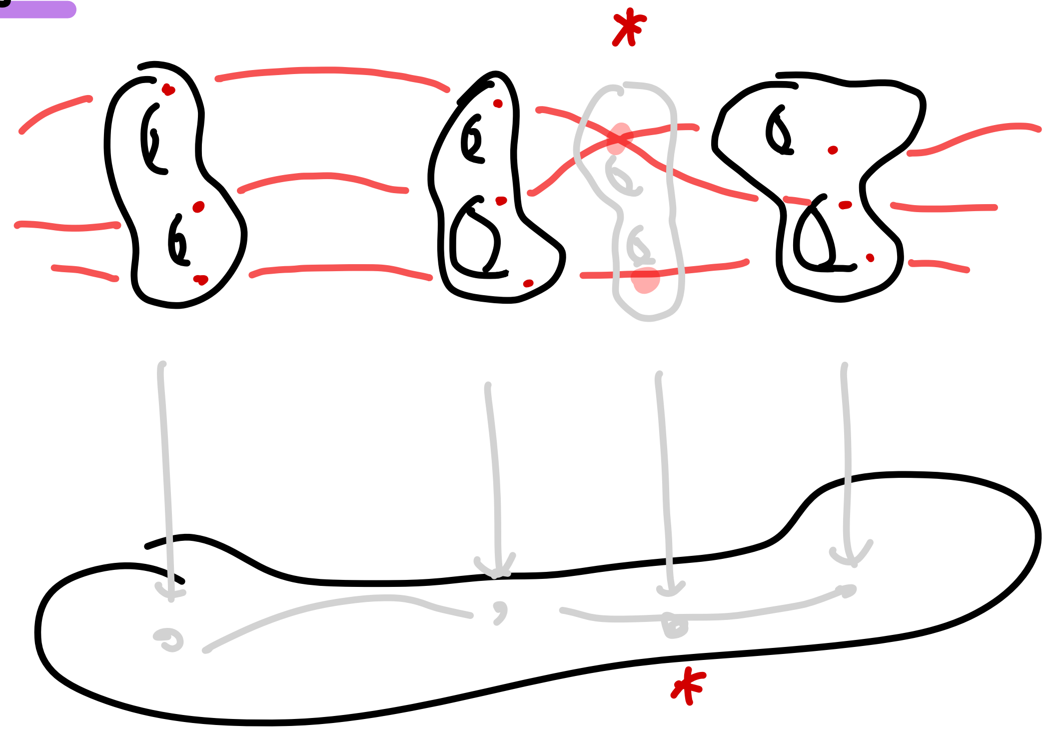
tropical

Def'n $x \in X$ is a Weierstrass point if

- $\exists$ hyperplane $H \subset \mathbb{P}^{g-1}$ with $H \cap X \geq g \cdot x$
- $\exists$ effective divisor $D \sim K$ with $D \geq g \cdot x$

Weierstrass weights

$$\begin{array}{c}\mathcal{C}_g\\ \downarrow\\ \mathcal{M}_g\end{array}$$

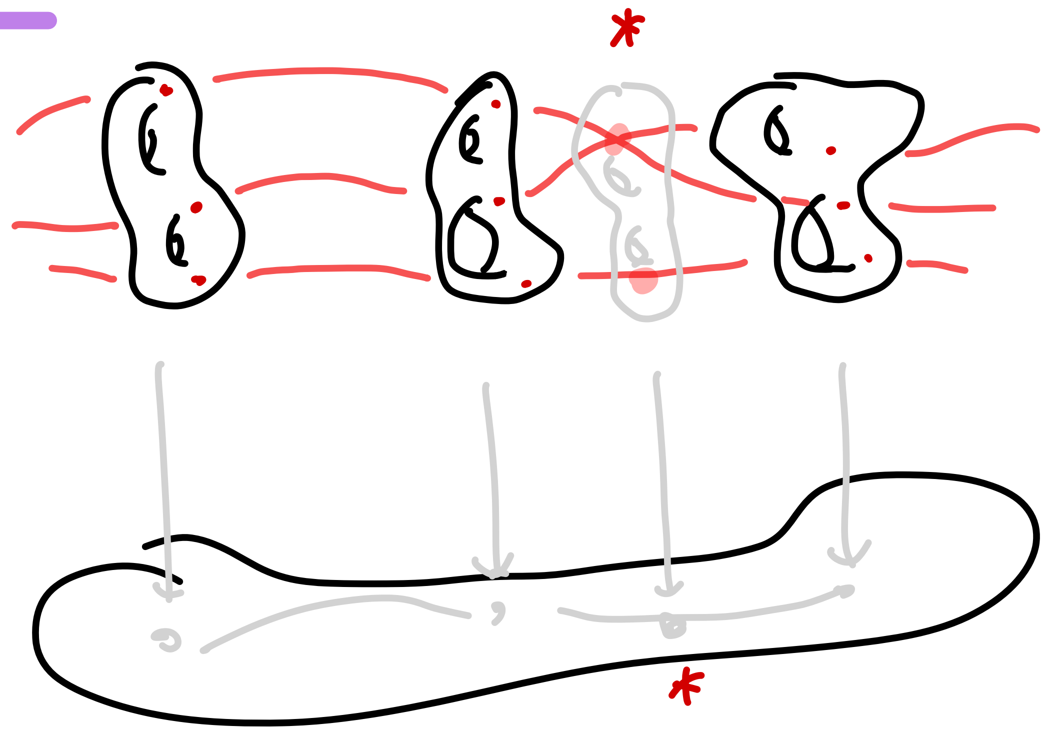


Sometimes Weierstrass points "collide".

Def'n (Informal) Weierstrass **weight** tracks # of W-pts on nearby generic curve.

Weierstrass weights

$$\mathcal{C}_g \to \mathcal{M}_g$$



Def'n. (Formal, sketch) Weierstrass weight of $x \in X$ is computed by taking derivatives* in the "Wronskian."

* NOT tropical :(

What are Weierstrass points?

algebraic curves $\xrightarrow{\text{Tropicalize!}}$ metric graphs

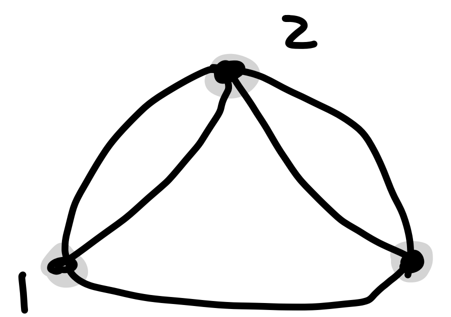
Tropical Weierstrass points

Γ a metric graph of genus g

Recall :

- canonical divisor $K = \sum_{x \in \Gamma} (\mathrm{val}(x) - 2)\, x$

Ex.



- $\mathrm{Rat}(K)$ = piecewise-linear f'ns on Γ whose poles are bounded by K

Tropical Weierstrass points

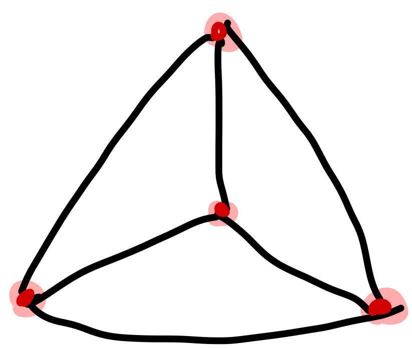
Γ a metric graph of genus g

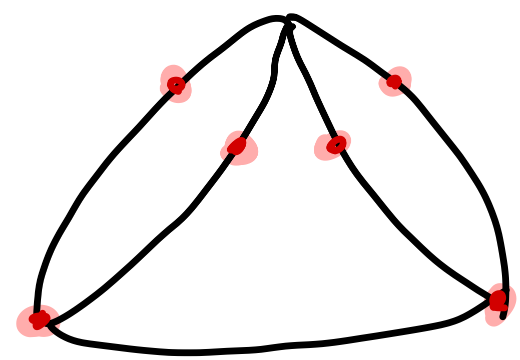
Def'n $x \in \Gamma$ is a (tropical) Weierstrass point if

- exists effective divisor $D \sim K$ with $D \geq g \cdot x$

$\Leftrightarrow$ • exists $f \in Rat(K)$ with $zeros(f) \geq g \cdot x$

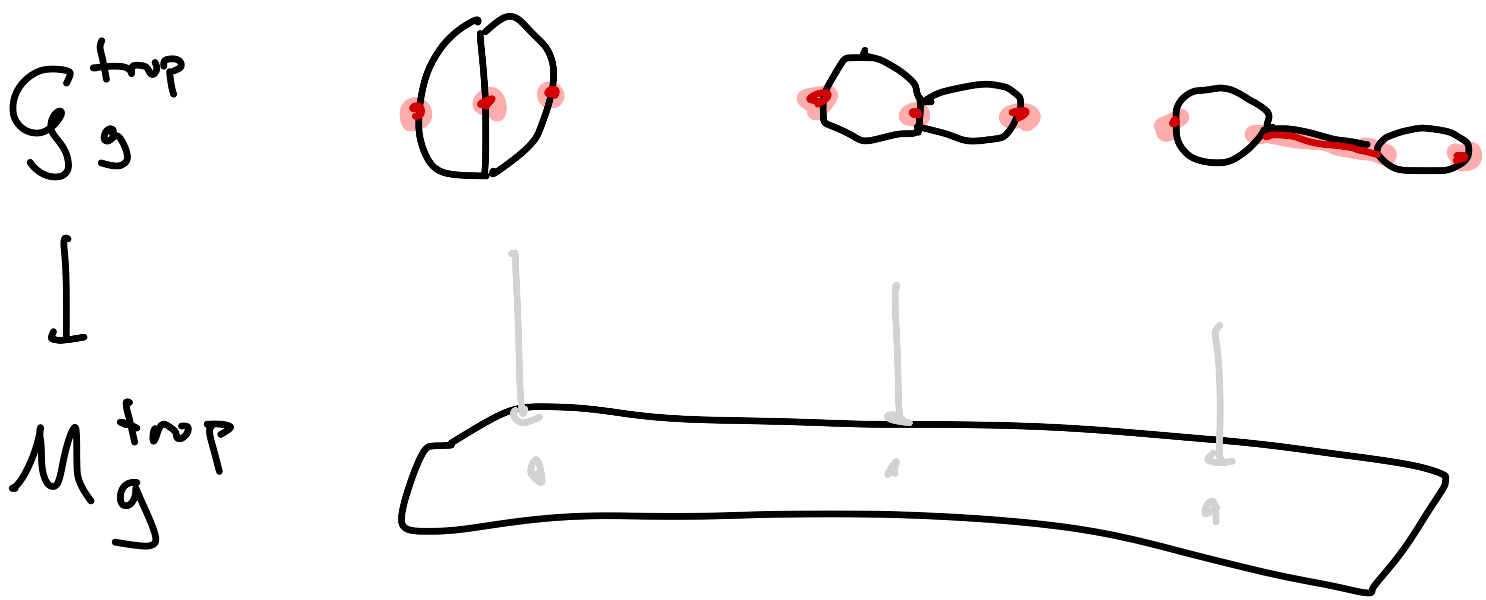
Ex.





Tropical Weierstrass weights

Problem. What is "good" notion of tropical weight?



Fact. Tropical Weierstrass locus can be infinite, containing positive-length segments

Tropical Weierstrass weights

Γ a metric graph of genus g

<u>Def'n.</u> (AGR) The Weierstrass weight of a closed, connected subset $A \subset \Gamma$ is

$$w(A) = (g+1)(g(A) - 1) - \sum_{v \in \partial A} (s_0^v(K) - 1)$$

where

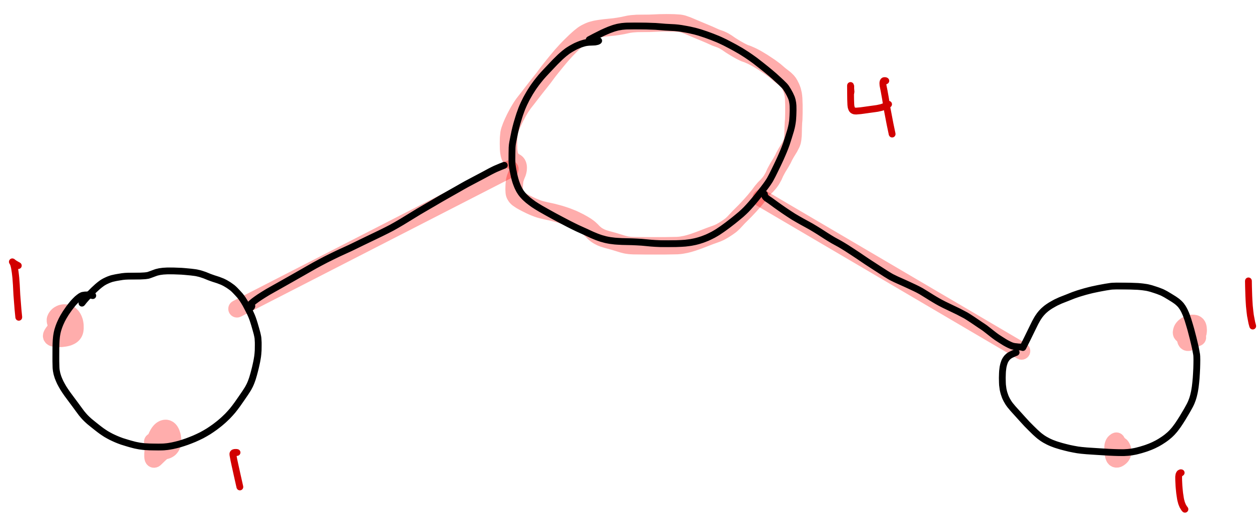
- g = genus of graph
- $g(A)$ = genus of subset
- ∂A = outgoing directions from A
- $s_0^v(K)$ = minimal slope along direction v in $\mathrm{Rat}(K)$

Tropical Weierstrass weights

compute via reduced divisors

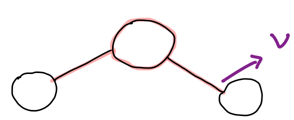
$$w(A) = (g+1)(g(A) - 1) - \sum_{v \in \partial A} (s_0^v(K) - 1)$$

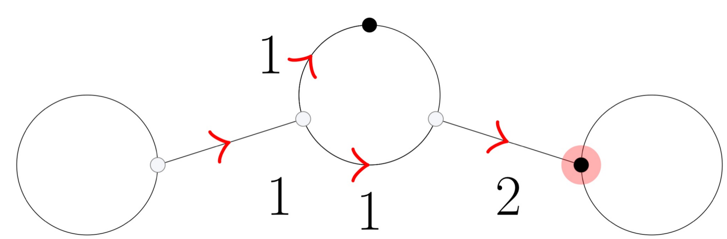
Ex.



- slope computation $\{s_i^v\} = \{0, 1, 2\}$

$v =$

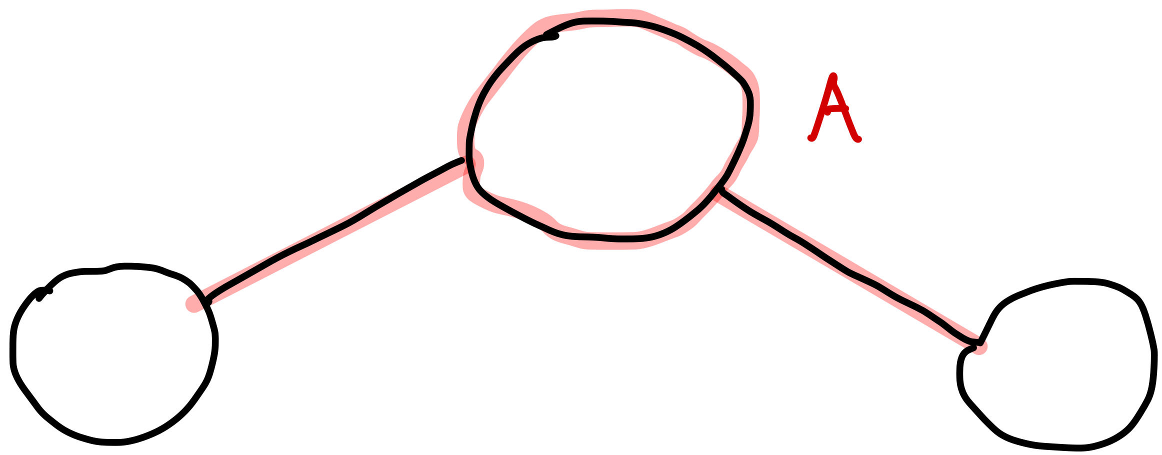




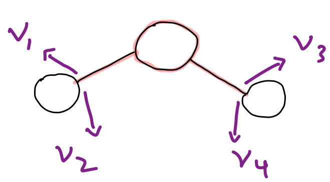
Tropical Weierstrass weights

$$w(A) = (g+1)\big(g(A) - 1\big) - \sum_{v \in \partial A} \big(s_0^v(K) - 1\big)$$

Ex.



$$w(A) = (3+1)(1-1) - \Big(\underbrace{(0-1) + \cdots}_{4 \text{ times}} \Big) = 4$$

- $\partial A =$ 

Tropical Weierstrass weights

$$w(A) = (g+1)(g(A)-1) - \sum_{v \in \partial A} (s_0^v(K) - 1)$$

Ex. If $A = \{x\}$, single point, then

$$w(x) = D_x(x) - g + 1$$

↳ reduced divisor

Ex. If $A = \Gamma$, $w(\Gamma) = (g+1)(g-1) = g^2 - 1$*

* Recall earlier: total weight $g^3 - g$

Tropical Weierstrass weights

$$w(A) = (g+1)(g(A)-1) - \sum_{v \in \partial A} (s_0^v(K) - 1)$$

Theorem (AGR) If X tropicalizes to Γ, then

$$w(A) = \frac{1}{g} \cdot \left(\begin{array}{c} \text{total Weierstrass weight of} \\ x \in X \text{ tropicalizing to } A \end{array} \right)$$

Cor. Total Weierstrass weight of $\text{trop}^{-1}(A)^*$ is a multiple of g.

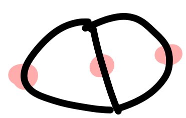
Cor. If A^* contains a cycle $(g(A) \geq 1)$ then at least one Weierstrass pt tropicalizes to A.

* If A is Lw-maxl

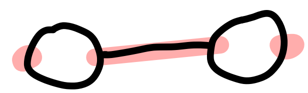
Open problems

- Can we characterize graphs where generic choice of edge lenghts $\Rightarrow$ $g^2 - 1$ distinct W-pts?

Ex.



✓ Yes

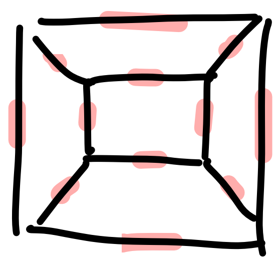


✗ No

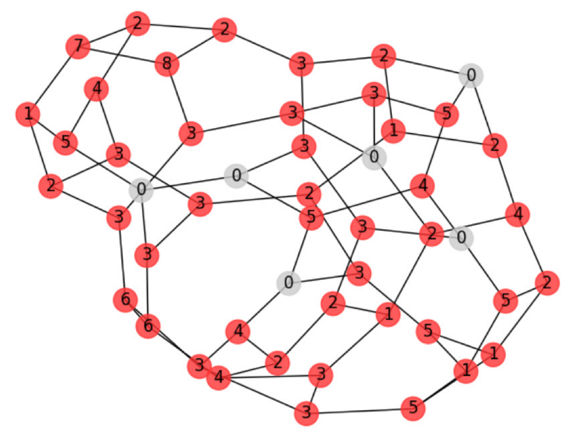
Open problems

- Can we characterize unit-edge-length graphs whose vertices are all non-Weierstrass pts?

Ex.

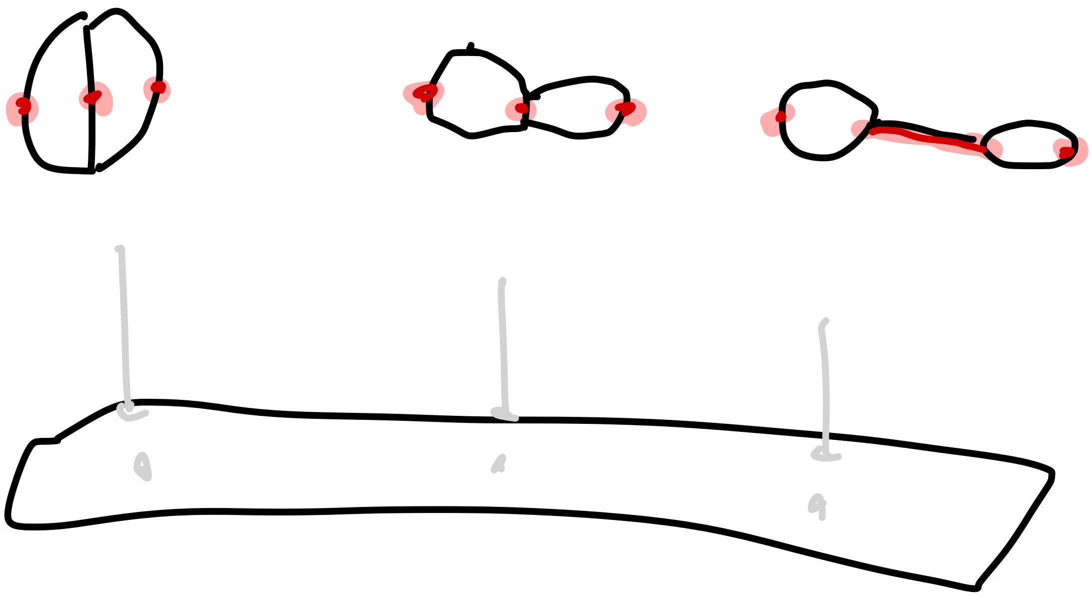


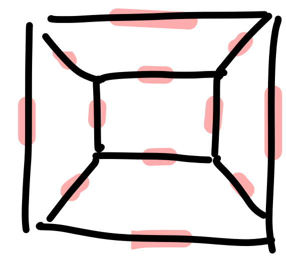
✓ Yes

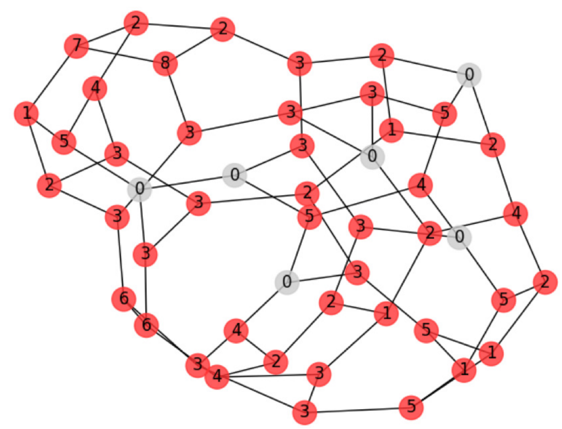


✗ No

Weights of tropical Weierstrass points







Thanks for listening!