

Weights of tropical

Weierstrass points

12 March 2025

Informal Tropical Geometry, Taipei

Harry Richman, NCTS



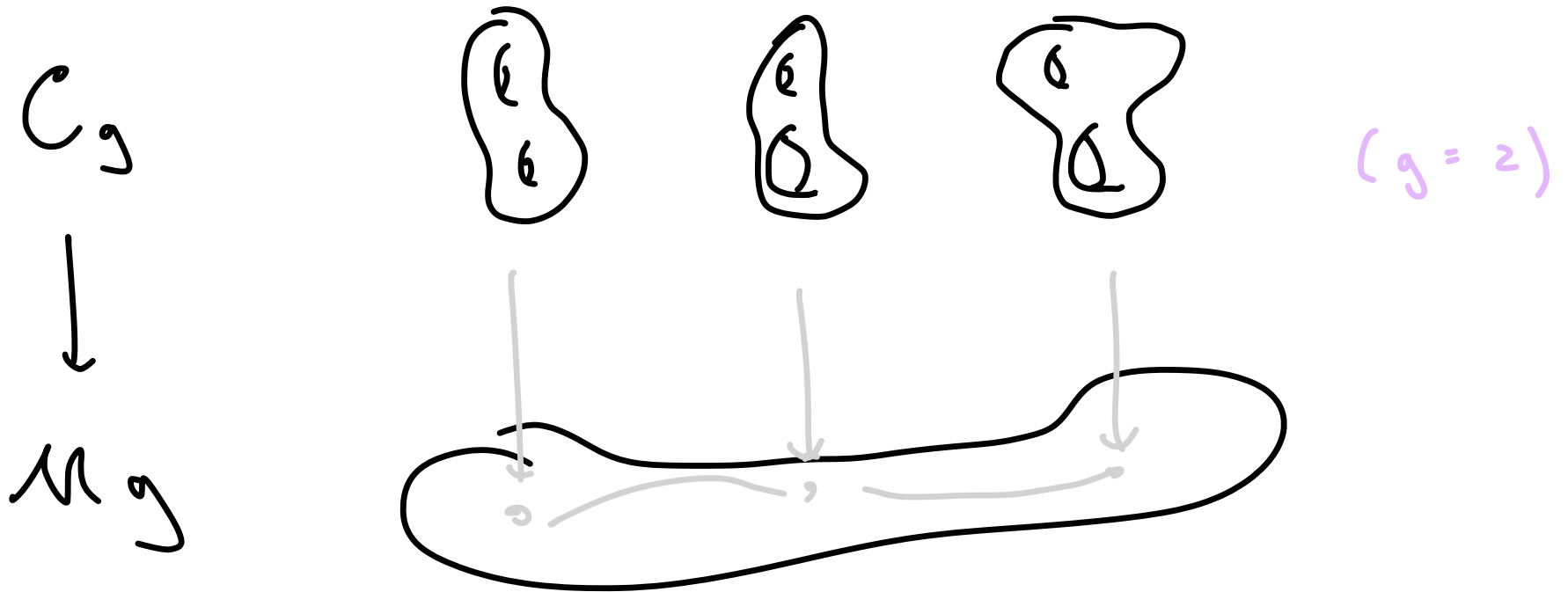
joint w/ Omid Amini & Lucas Gierczak

École Polytechnique

Marseille

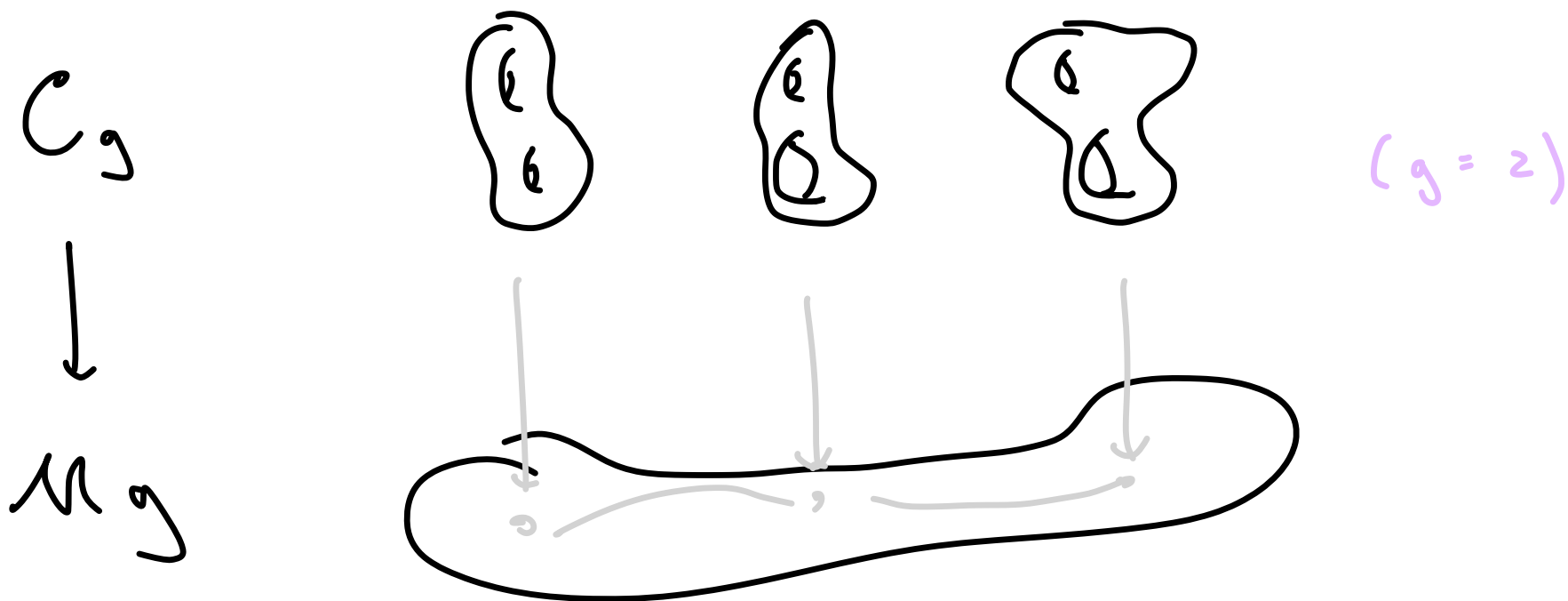
Why Weierstrass points?

Goal: understand moduli space of algebraic curves \mathcal{M}_g



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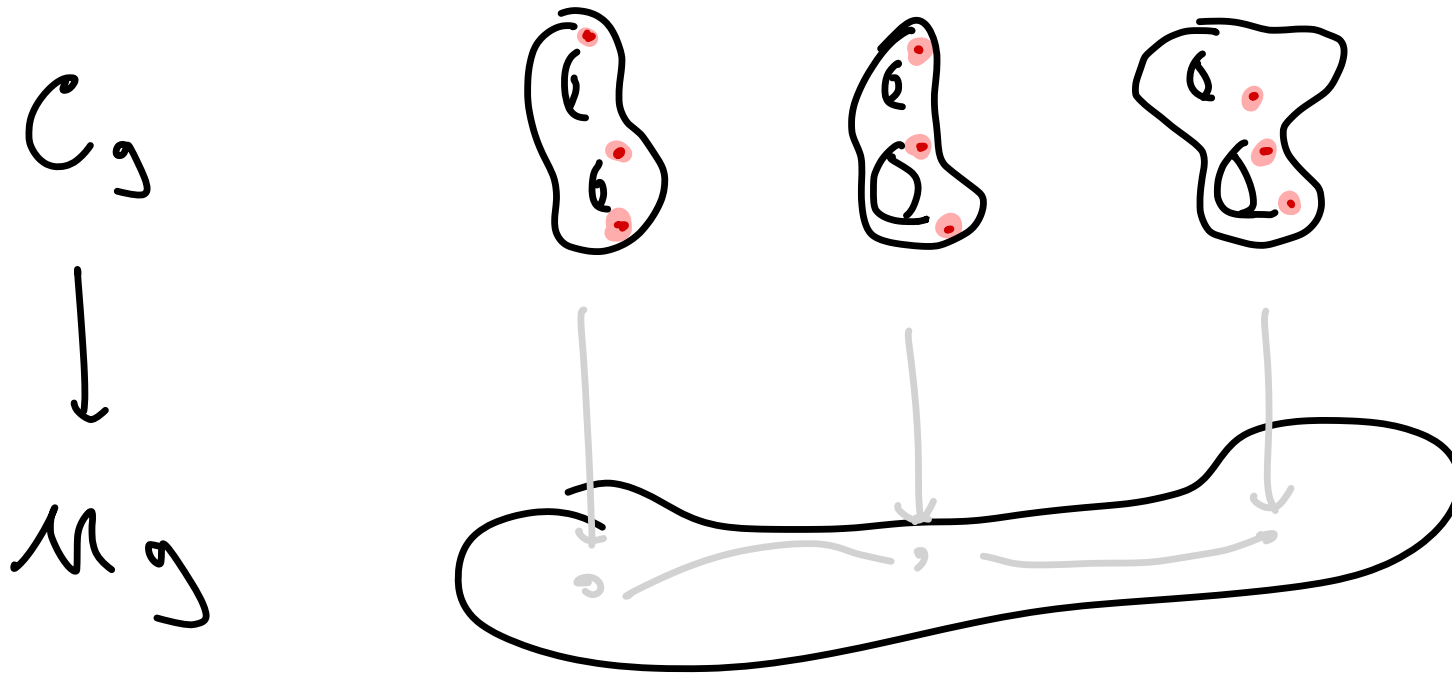
• Homology groups $H_k(\mathcal{M}_g)$?

• Rational parametrization: $\exists \mathbb{A}^N \rightarrow \mathcal{M}_g$?

• $\exists X \subset \mathcal{M}_g$
complete subvariety
of given dim?

Why Weierstrass points?

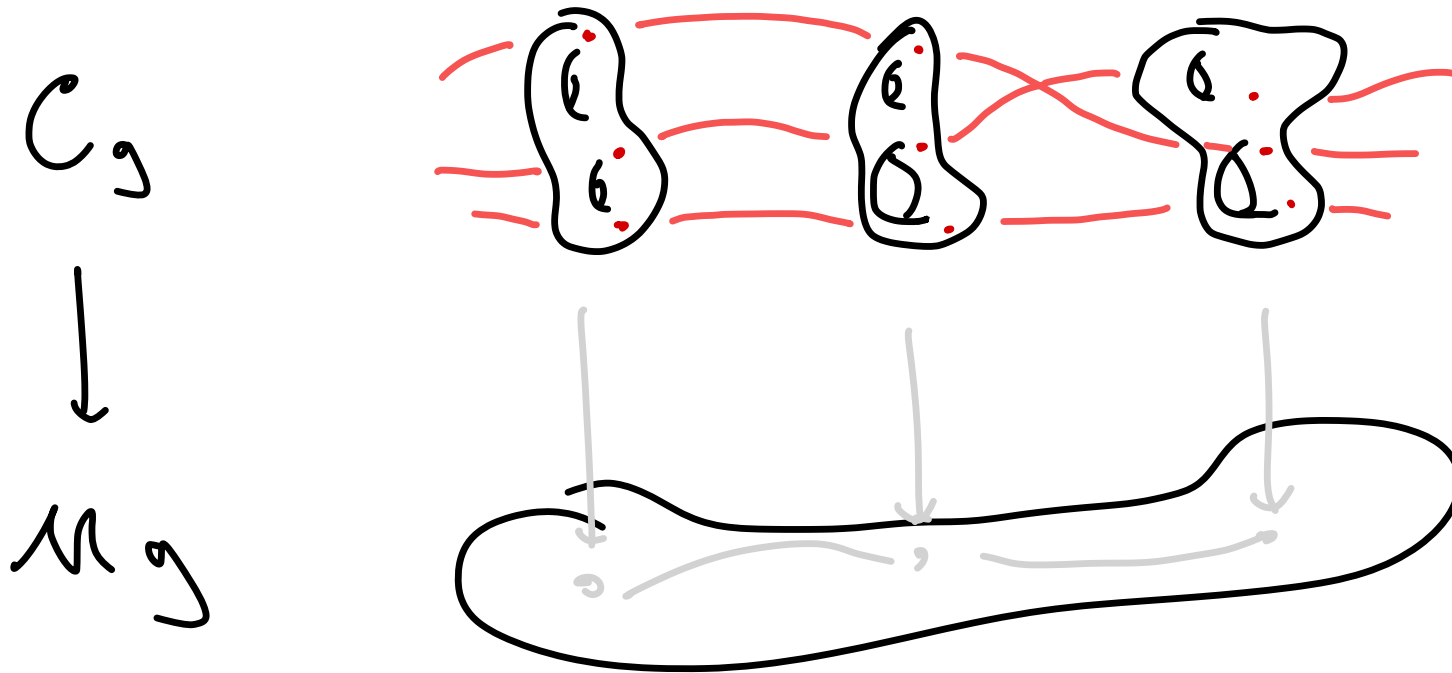
Goal: understand moduli space of algebraic curves \mathcal{M}_g



Slogan: "Weierstrass pts are coordinates on \mathcal{M}_g "

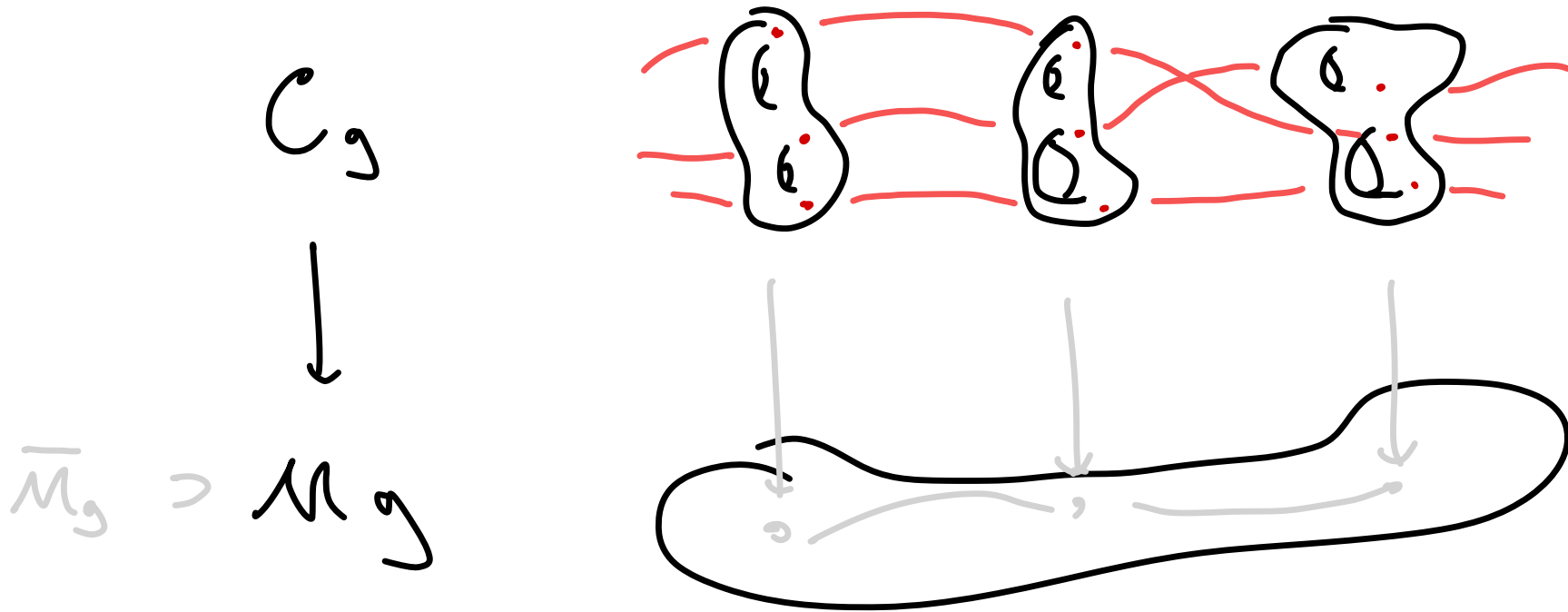
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Goal: understand moduli space of algebraic curves \mathcal{M}_g



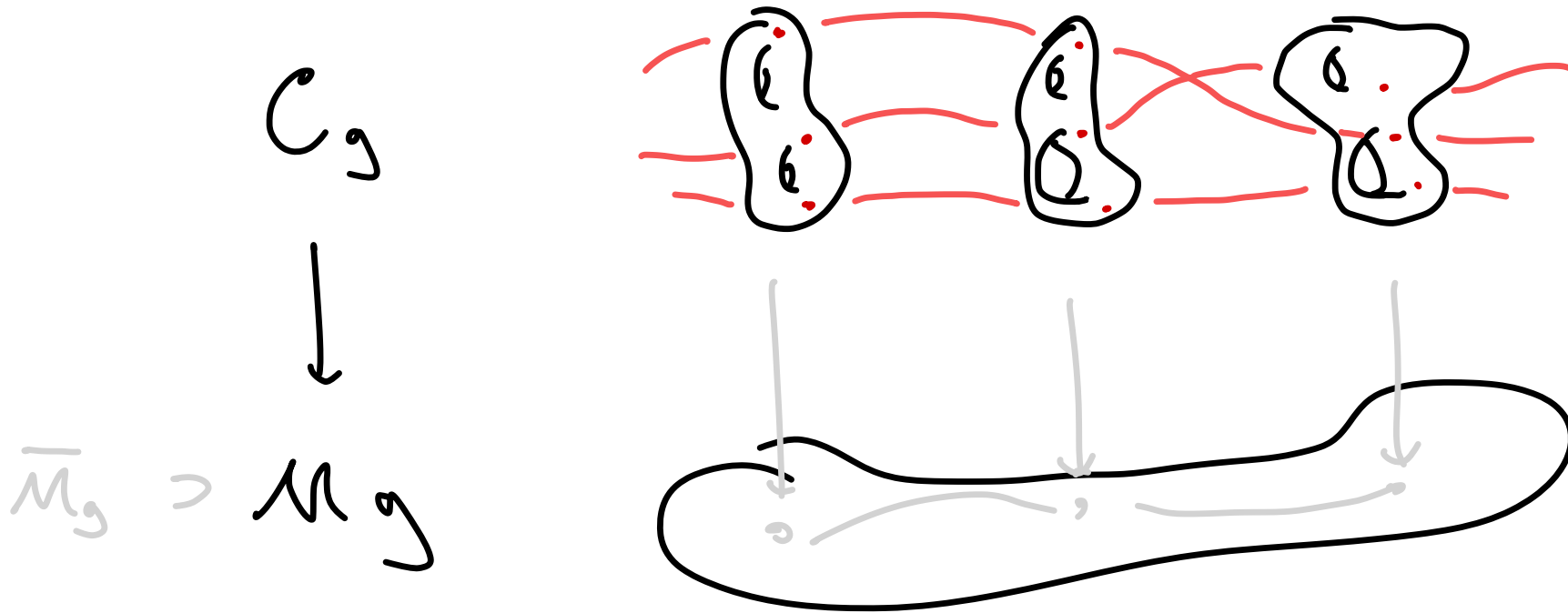
Slogan: "Weierstrass pts are coordinates on \mathcal{M}_g "

Why Weierstrass points?



Problem: How do Weierstrass points behave
on boundary of $\bar{\mathcal{M}}_g$?

Why Weierstrass points?

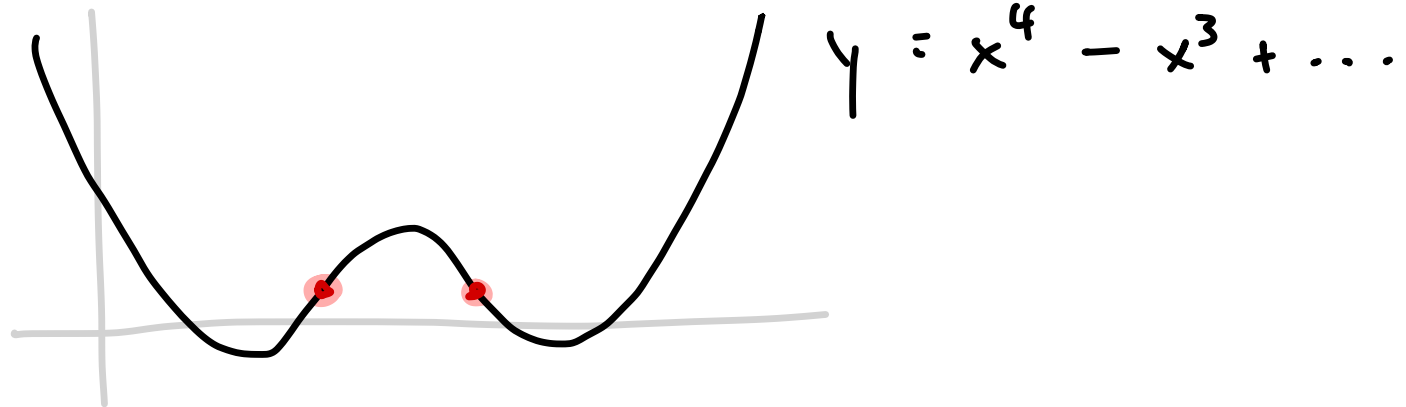


Problem: How do Weierstrass points behave
on boundary of $\bar{\mathcal{M}}_g$?

What are Weierstrass points? (Background)

Recall: first-year calculus, inflection point of a graph

Ex.

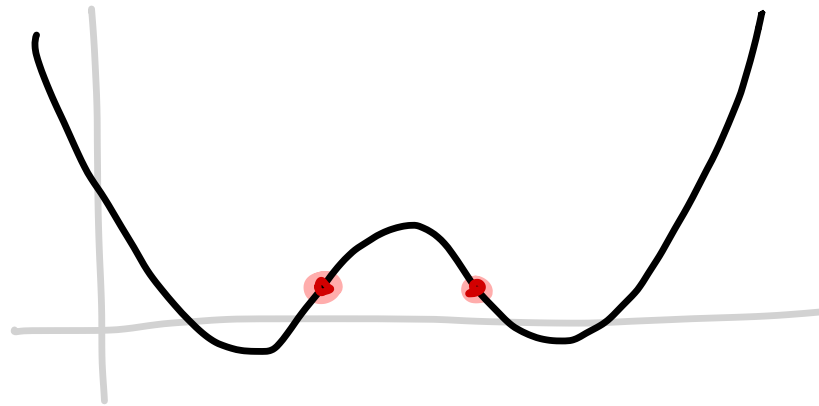


- Inflection?
- "Concavity changes sign"
 - tangent line intersects graph w/
multiplicity ≥ 3

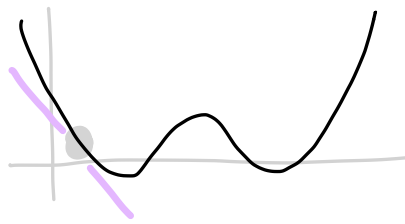
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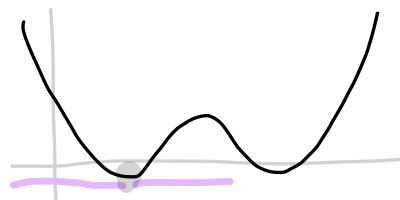
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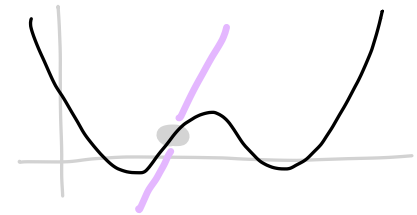
$$y = x^4 - x^3 + \dots$$



X



X



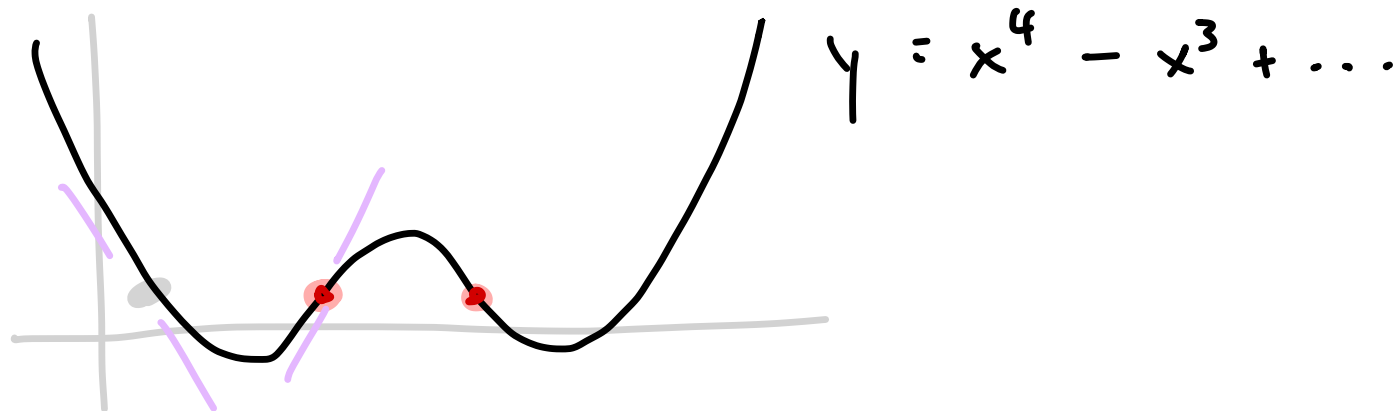
✓

Inflection?

What are Weierstrass points?

Recall: first-year calculus, inflection point of a graph

Ex.



Fact: If polynomial $f(x)$ has degree d ,
 $y = f(x)$ has $d - 2$ * inflection pts

* counted w/ multiplicity

What are Weierstrass points?

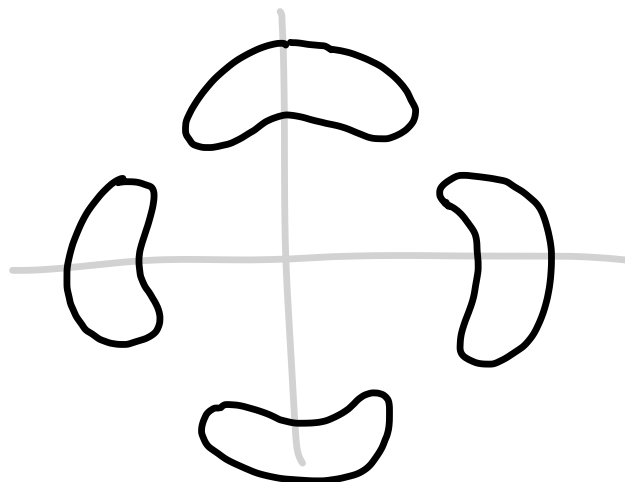
X an alg. curve of genus $g \geq 2$

Def'n. Weierstrass points of X are

inflection pts of canonical embedding $X \rightarrow \mathbb{P}^{g-1}$

\mathbb{F}_x .

$g = 3$



$$144(x^4 + y^4) - 225(x^2 + y^2)$$

$$+ 350x^2y^2 + 81 = 0$$

"Trott curve"

via Wikipedia

Q: How many inflection pts?

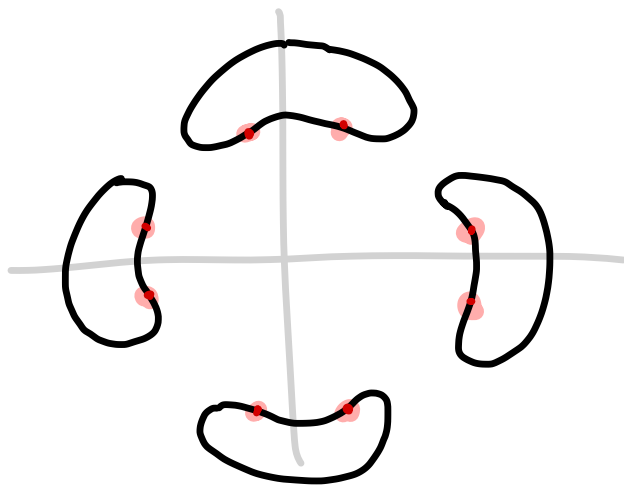
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Ex.



$$144(x^4 + y^4) - 225(x^2 + y^2) + 350x^2y^2 + 81 = 0$$

Fact: generic alg. curve has $g^3 - g$ distinct
Weierstrass pts

What are Weierstrass points?

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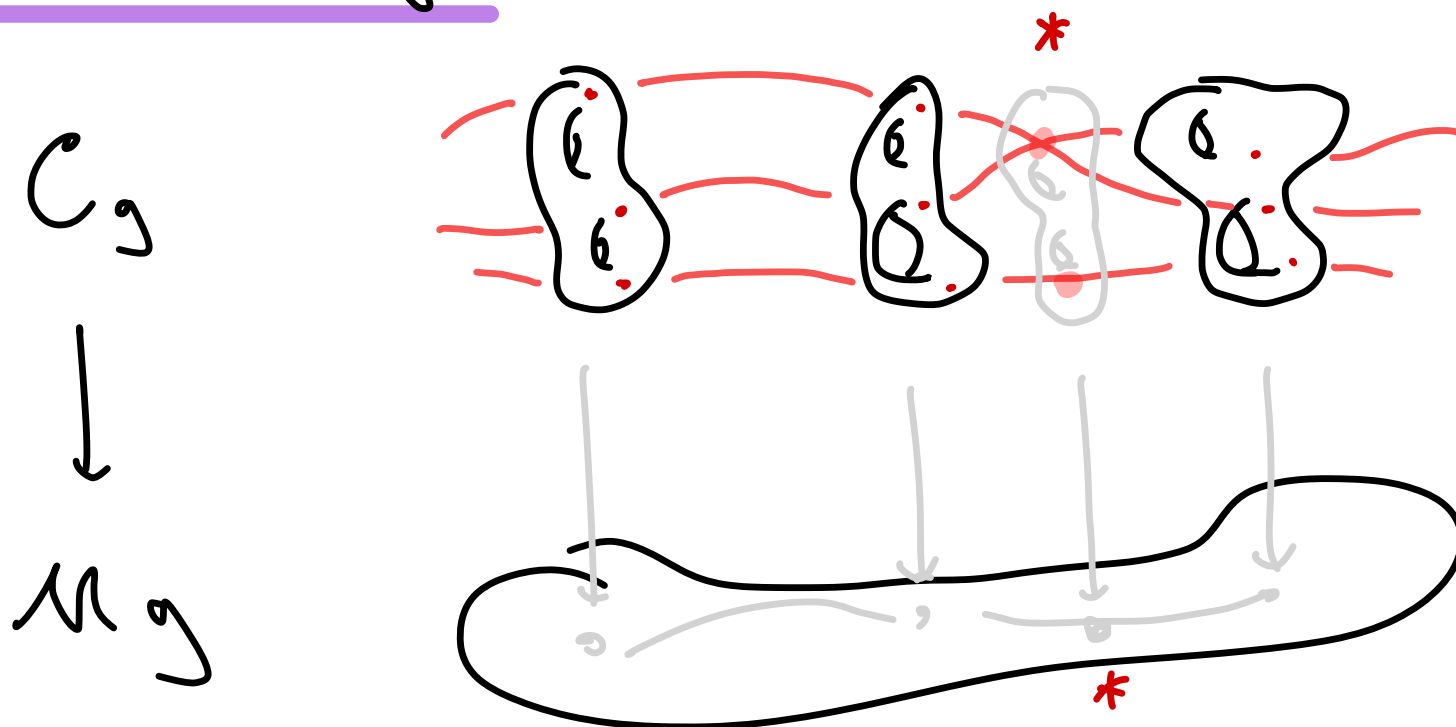
Divisor theory

 Def'n $x \in X$ is a Weierstrass point if

tropical

- \exists hyperplane $H \subset \mathbb{P}^{g-1}$ with $H \cap X \geq g \cdot x$
- \exists effective divisor $D \sim K$ with $D \geq g \cdot x$

Weierstrass weights



Sometimes Weierstrass points "collide".

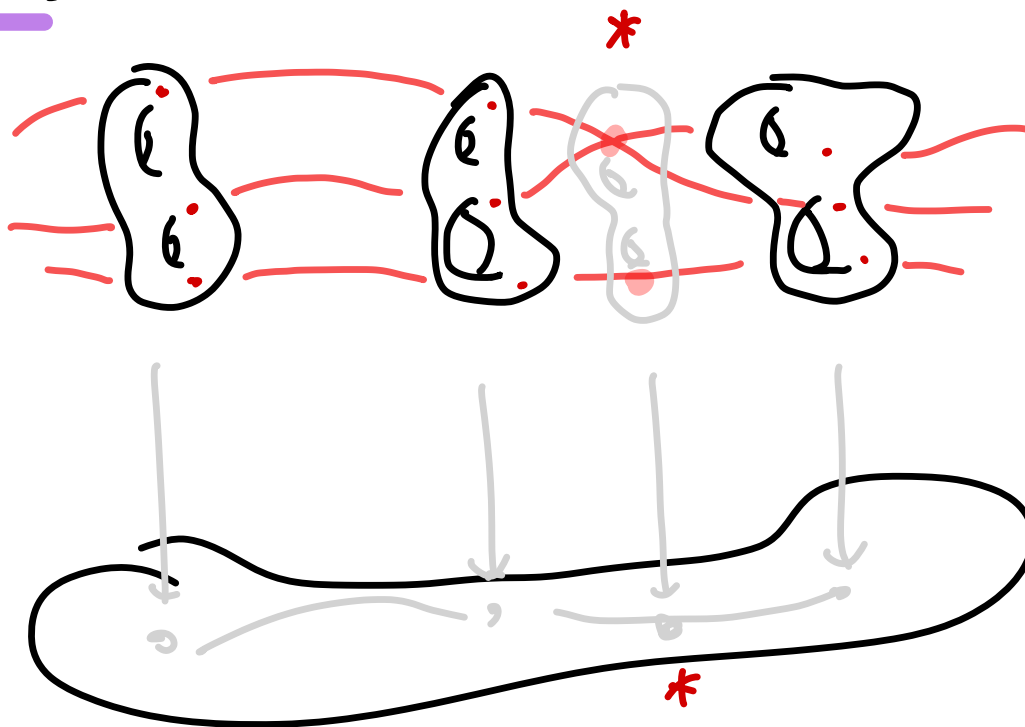
Def'n. (Informal) Weierstrass weight tracks # of
W-pts on nearby generic curve.

Weierstrass weights

C_g



M_g



Def'n. (Formal, sketch) Weierstrass weight of $x \in X$
is computed by taking derivatives^{*} in the
"Wronskian."

* NOT tropical :-

What are Weierstrass points?

algebraic
curves

Tropicalize!



metric
graphs

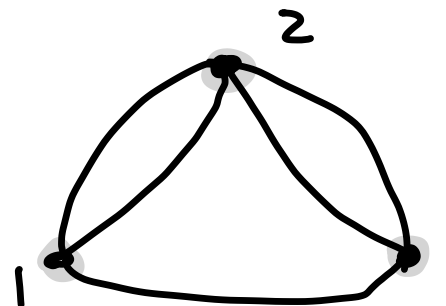
Tropical Weierstrass points

Γ a metric graph of genus g

Recall :

- canonical divisor $K = \sum_{x \in \Gamma} (\text{val}(x) - 2) x$

Ex.



- Rat(K) = piecewise-linear f'ns on Γ whose poles are bounded by K

Tropical Weierstrass points

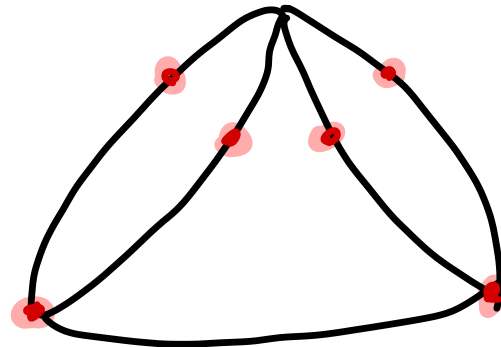
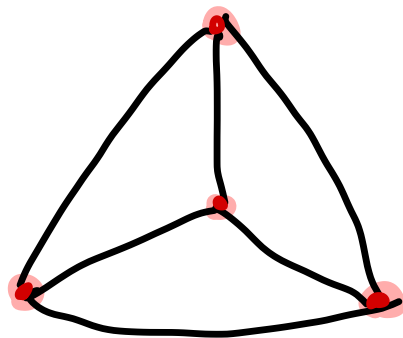
Γ a metric graph of genus g

Def'n $x \in \Gamma$ is a (tropical) Weierstrass point if

• exists effective divisor $D \sim K$ with $D \geq g \cdot x$

\Leftrightarrow • exists $f \in \text{Rat}(K)$ with $\text{zeros}(f) \geq g \cdot x$

Ex.



Tropical Weierstrass points

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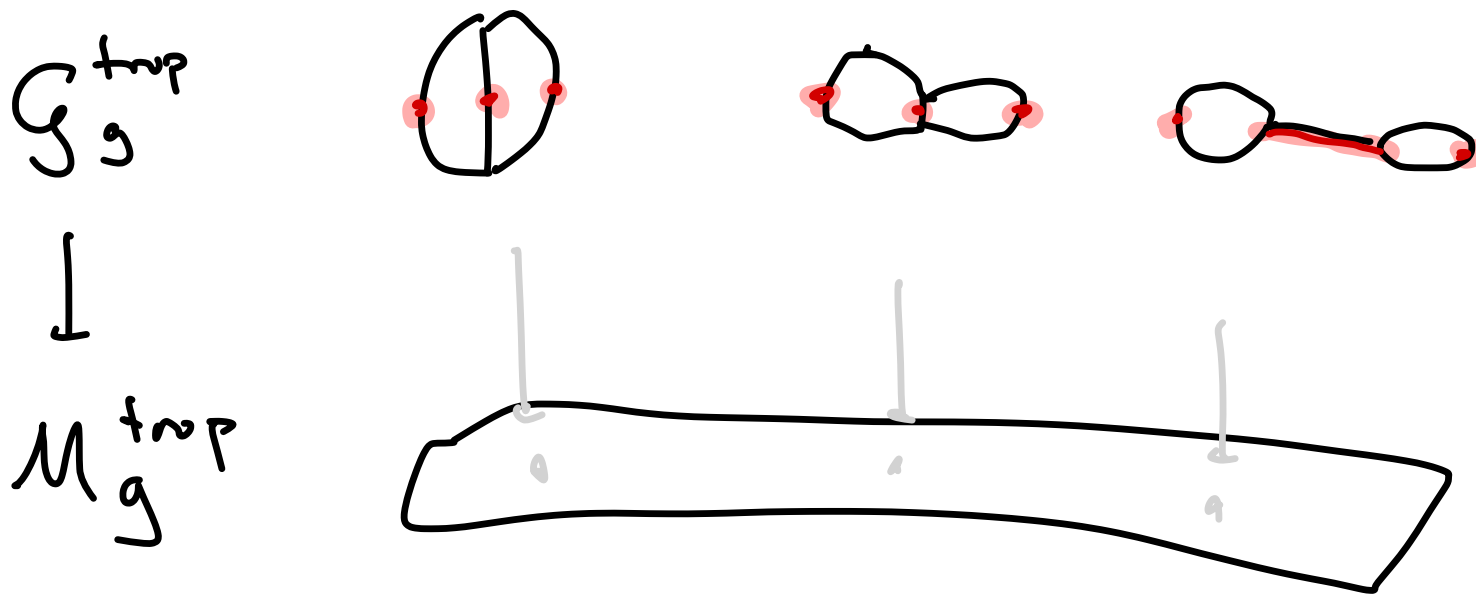
\Leftrightarrow • exists $f \in \text{Rat}(K)$ with $\text{zeros}(f) \geq g \cdot x$

Ex.

[more examples on
Computer]

Tropical Weierstrass weights

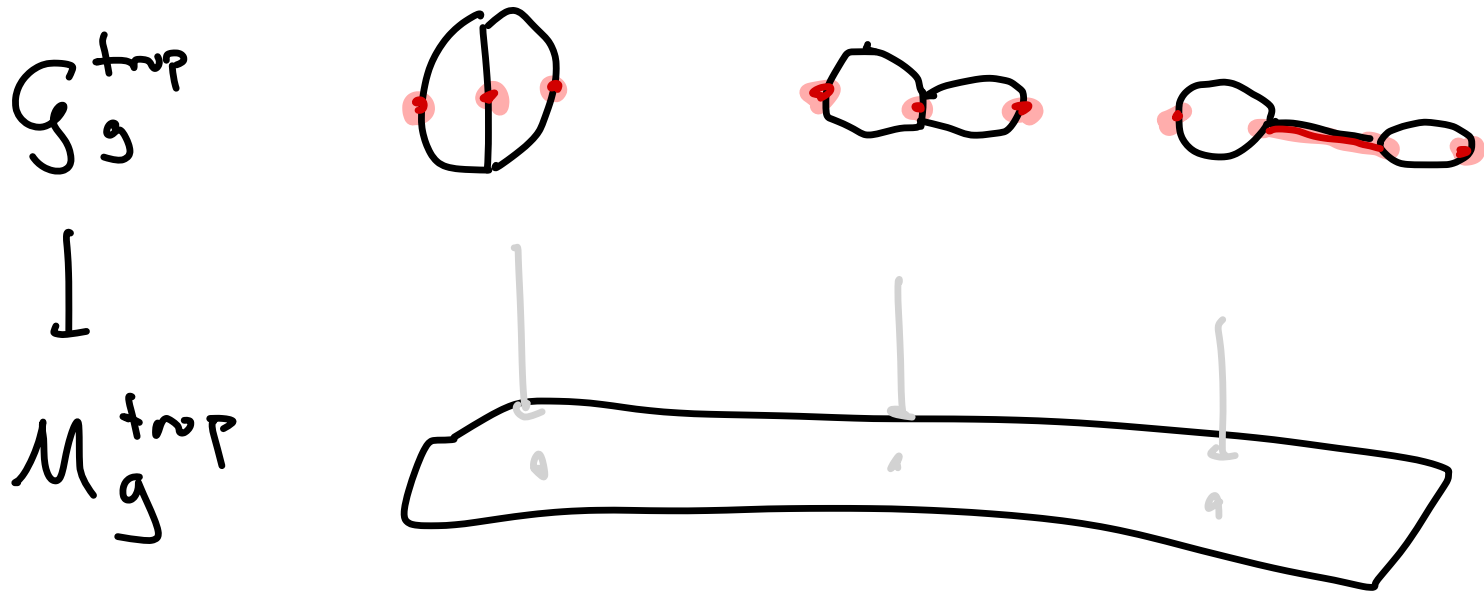
Problem. What is "good" notion of tropical weight?



Fact. Tropical Weierstrass locus can be infinite, containing positive-length segments

Tropical Weierstrass weights

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Fact. Tropical Weierstrass locus can be infinite, containing positive-length segments

Tropical Weierstrass weights

Γ a metric graph of genus g

Def'n. (AGR) The Weierstrass weight of a closed, connected subset $A \subset \Gamma$ is

$$w(A) = \underbrace{(g+1)}_{\text{yellow}} \left(\underbrace{g(A)}_{\text{orange}} - 1 \right) - \sum_{v \in \underbrace{\partial A}_{\text{green}}} \left(\underbrace{s_0^v(K)}_{\text{cyan}} - 1 \right)$$

where

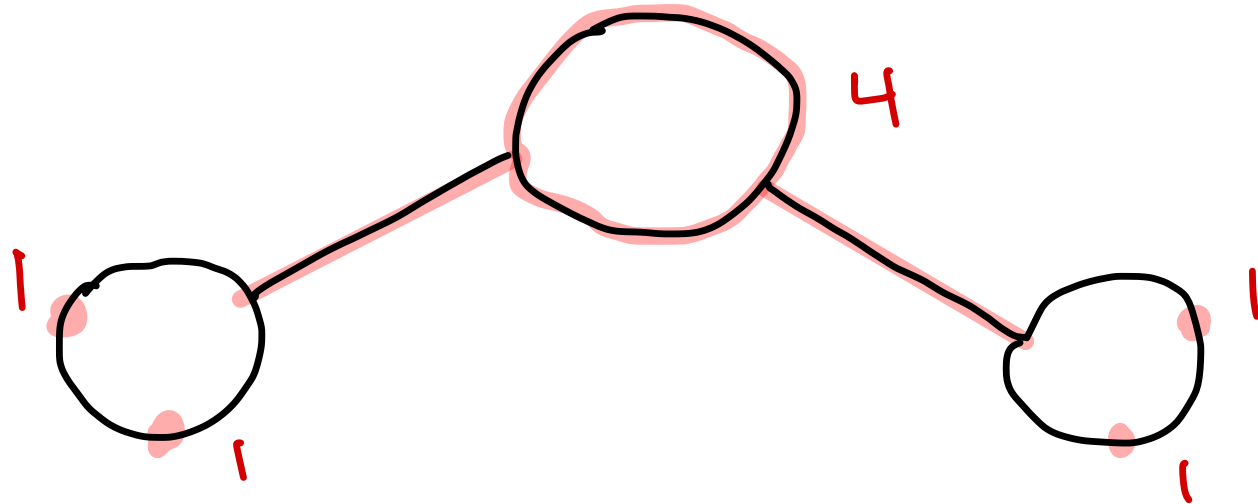
- $\underbrace{g}_{\text{yellow}}$ = genus of graph
- $\underbrace{g(A)}_{\text{orange}}$ = genus of subset
- $\underbrace{\partial A}_{\text{green}}$ = outgoing directions from A
- $\underbrace{s_0^v(K)}_{\text{cyan}}$ = minimal slope along direction v in $\text{Rat}(K)$

Tropical Weierstrass weights

compute via
reduced divisors

$$w(A) = \underline{(g+1)} (\underline{g(A)} - 1) - \sum_{v \in \partial A} (\underline{s_0^v(K)} - 1)$$

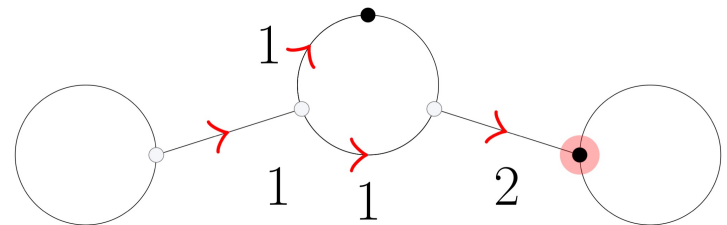
Ex.



• slope computation

$$\{ \underline{s_i^v} \} = \{ \underline{0, 1, 2} \}$$

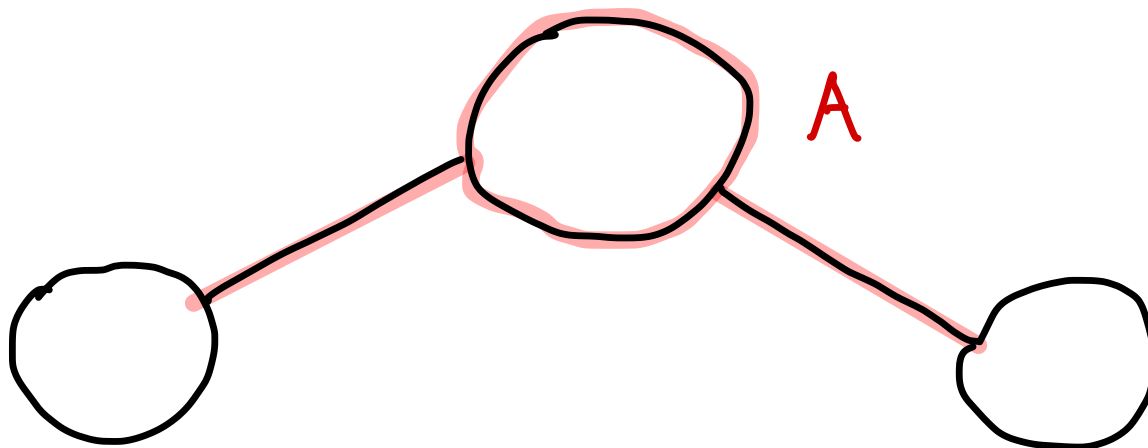
$v =$



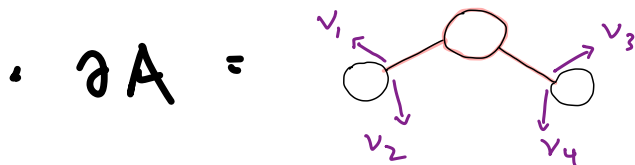
Tropical Weierstrass weights

$$w(A) = \underline{(g+1)} (\underline{g(A)} - 1) - \sum_{v \in \partial A} (\underline{s_v^0(K)} - 1)$$

Ex.



$$w(A) = \underline{(3+1)} (\underline{1} - 1) - \underbrace{((\underline{0} - 1) + \dots)}_{4 \text{ times}} = 4$$



Tropical Weierstrass weights

$$w(A) = (g+1)(g(A)-1) - \sum_{v \in \partial A} (s_v^0(K)-1)$$

Ex. If $A = \{x\}$, single point, then

$$w(x) = D_x(x) - g + 1$$

↳ reduced divisor

Ex. If $A = \Gamma$, $w(\Gamma) = (g+1)(g-1) = g^2 - 1^*$

* Recall earlier: total weight $g^3 - g$

Tropical Weierstrass weights $w(A) = (g+1)(g(A)-1) - \sum_{v \in \partial A} (s_v^0(K)-1)$

Theorem (AGR) If X tropicalizes to Γ , then

$$w(A) = \frac{1}{g} \cdot \left(\begin{array}{l} \text{total Weierstrass weight of} \\ x \in X \text{ tropicalizing to } A \end{array} \right)$$

Cor. Total Weierstrass weight of $\text{trop}^*(A)^*$ is
a multiple of g .

Cor. If A^* contains a cycle ($g(A) \geq 1$) then at
least one Weierstrass pt tropicalizes to A .

* If A is L_w -max

Open problems

- Can we characterize graphs where generic choice of edge lengths $\Rightarrow g^2 - 1$ distinct W-pts?

Ex.



✓ Yes

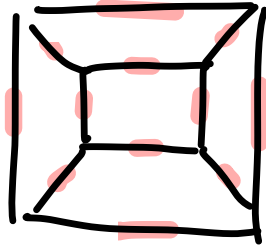


✗ No

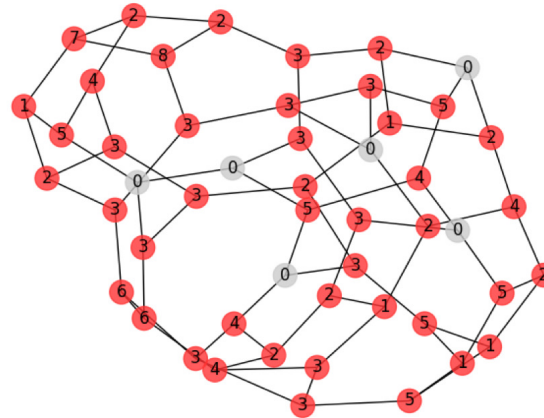
Open problems

- Can we characterize unit-edge-length graphs whose vertices are all non-Weierstrass pts?

Ex.



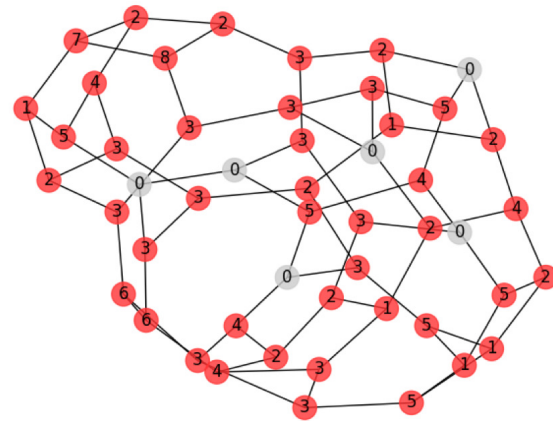
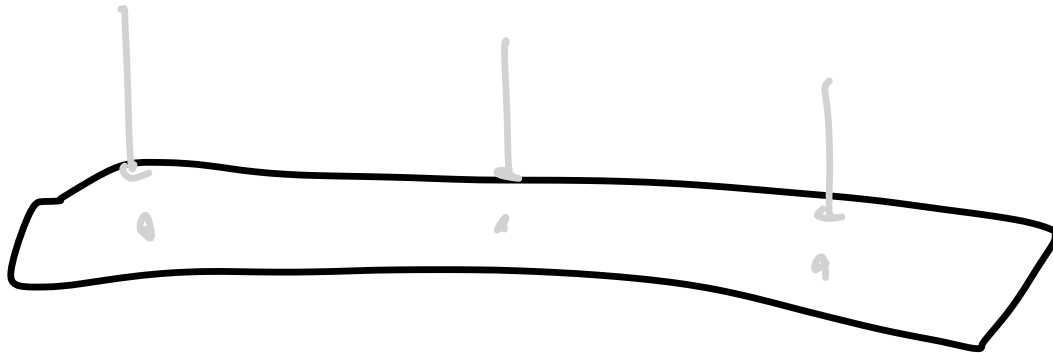
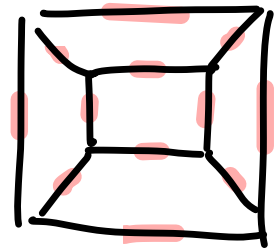
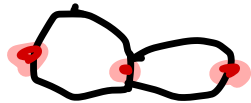
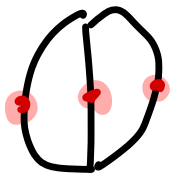
✓
Yes



✗
No

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Thanks for listening !

